Stability for determining the principal coefficient of parabolic equation

Ali Demir, Arzu Erdem

Department of Mathematics, Kocaeli University, Umuttepe Kampusu, 41380 Izmit-Kocaeli, Turkey

Abstract

This work investigates stability of an inverse problem for a backward linear parabolic heat problem by using an initial temperature measurement. This kinds of problem has been widely used in a various field of pure and applied science. The necessary condition of the minimizer for the cost functional are constructed by using the optimal control theory. Stability of the minimizer for the cost functional is proved based on the necessary condition.

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1. Introduction

In this paper we deal with the stability for the inverse problem of determining the principal coefficient $c(t)$ satisfying the linear parabolic heat equation

$$u_t = u_{xx} - c(t)u + f(x, t), \quad x \in (0, l), \quad t \in (0, T] \tag{1}$$

along with the terminal time condition

$$u(x, T) = \phi(x) \quad x \in [0, l] \tag{2}$$

the boundary conditions

$$u_t(0, t) = u(l, t) = 0 \quad t \in [0, T] \tag{3}$$

and the additional condition

$$u(x, 0) = g(x) \quad x \in [0, l], \tag{4}$$

where $f(x, t), \phi(x), g(x)$ are all continuously differentiable. Problems of this type are some active area of research, such as heat conduction, optical medical imaging, geophysics of exploration. When the function $c(t)$ is given the problem (1)–(3) has been analyzed as a direct problem in [4]. However, the identification of time dependent coefficient $c(t)$ in (1)–(4) which is referred as the inverse problem lead to ill-posed or improperly posed problem in the sense of Hadamard [1,6,7,11]. Inverse coefficient problem has already been studied by several authors.

Choulli and Yamamoto [2] proved, under some conditions, the inverse problem of recovering the coefficient $q(x)$, appearing in an initial-boundary value problem for the equation $u_t = \Delta u + q(x)u$ from overdetermined final data is locally well-posed in $L^2$ around 0 when $q$ is assumed to be a priori supported in some suitable subset. Shamsi and Dehghan [10] proposed a Legendre pseudospectral method for solving approximately an inverse problem of determining an unknown control parameter $p(t)$ in the semilinear time-dependent three-dimensional diffusion equation $u_t = \Delta u + p(t)u + k(x, t)$ subject to initial condition, Dirichlet boundary conditions and the integral overspecification over the spacial domain. Trucu et al.
[12] discussed the retrieval of the time-dependent coefficient along with initial boundary value problem from various types of measured noisy and exact data. Pourgholi et al. [8] have suggested the numerical method combining the use of the finite difference method with the solution of ordinary differential equation for the determination of unknown coefficient in an inverse heat conduction problem. Fernandez and Pola [5] have studied the uniqueness of the inverse coefficient problem under appropriate assumptions.

Egger et al. [3] derived global uniqueness of a solution \((u, q)\) the inverse problem of determining the function \(q(\cdot)\) for the equation \(-u_t(x, t) + u_x + q(u) = f(x, t)\) with the initial condition and Dirichlet boundary conditions and Hölder stability of the functions \(u\) and \(q\) with respect to errors in the measurements of the Neumann boundary data the initial condition and the a priori knowledge of the function \(q\). However, this method requires, assumptions on the functions \(q\), boundary conditions and solution \(u\).

In this paper, we use optimization technique to deal with backward-inverse heat conduction problem with the initial time observation. These method works by minimizing an objective function such that its minimum corresponds to the ideal design configuration. The optimization methods replace the ill-conditioned problem with a well-posed problem that must be solved repetitively through a systematic approach to an optimum solution. Yang et al. [13] investigated an inverse problem of determining the principal coefficient \(c(t)\) with the final time condition, Neumann–Dirichlet boundary data and initial time observation. In this work, first by using a transformation the backward problem (1)–(4) is reduced into the identification of source function in backward problem, namely, the inverse problem of determining the function \(q(\cdot)\) for the equation \(-u_t(x, t) + u_x + q(u) = f(x, t)\) with the initial condition and Dirichlet boundary conditions and Hölder stability of the functions \(u\) and \(q\) with respect to errors in the measurements of the Neumann boundary data the initial condition and the a priori knowledge of the function \(q\). However, this method requires, assumptions on the functions \(q\), boundary conditions and solution \(u\).

The method begins with using the following transformation:

\[
v(x, t) = u(x, t) r(t),
\]

where

\[
r(t) = \exp \left( \int_0^t c(\eta) d\eta \right).
\]

Transformations (5), (6) changes the problem (1)–(4) into the identification of source function in backward problem, namely

\[
u_1 = v_x + r(t)f(x, t), \quad x \in (0, l), \quad t \in (0, T],
\]

\[
u(x, T) = \varphi(x) r(T), \quad x \in [0, l],
\]

\[
u_x(0, t) = \nu(l, t) = 0, \quad t \in [0, T],
\]

\[
u(x, 0) = g(x) \quad x \in [0, l].
\]

The proof of the solvability of the inverse problem (7)–(10) in spaces \(C^{k+\beta}, \alpha \text{ fixed in } (0, 1)\) and \(k \in N\), is a continuous functions with Hölder continuous derivatives, has been constructed in [9], as follows.

**Theorem 1.** If \(g(x), \varphi(x) \in C^{2+k+\beta}[0, 1]\), \(g(x) \geq 0\), and the compatibility conditions up to first order are satisfied, then there exists a unique solution \(v \in C^{2+k+\beta+2\beta}[0, 1] \times [0, T]\), \(c \in C^{k+\beta}[0, T]\) of the inverse problem (7)–(10) which is continuously dependent upon data.

We now introduce the following optimal control problem. The optimization-based formulation of this inverse problem consists of the minimization of the objective functional

\[
J(\bar{r}) = \min_{r \in R_0} J(r),
\]

where

\[
J(r) = \frac{1}{2} \int_0^T |v(x, 0; r) - g(x)|^2 dx + \frac{\alpha}{2} \int_0^T r^2(t) dt,
\]

\(R_0 := \{ r(t) \in C[0, T] : r(T) = c_0, \text{ constant} \} \).

Here \(v(x, 0; r)\) denotes the initial solution of the problem (7)–(9) for given \(r(t)\).
3. Necessary condition of optimal control problem

For the solution of the optimization (11), (12) we derive the necessary condition.

**Theorem 2.** Let \( r(t) \) be the solution of the optimal control problem (11), (12) and \( \nu(x, t) \) be the solution of (7)–(9) corresponding to this optimal coefficient. Then for any \( h(t) \in R_0 \) the following integral inequality holds:

\[
\int_0^1 r(x, 0) - g(x)\zeta(x, 0) \, dx + \alpha \int_0^T r(t)[h(t) - r(t)] \, dt \geq 0.
\]

(13)

Here, \( \zeta(x, t) \) is the solution of the problem given by

\[
\begin{cases}
\xi_v = \xi_{xx} + (h(t) - r(t)) f(x, t), & x \in (0, 1), \ t \in (0, T] \\
\xi(x, T) = 0, & x \in [0, 1] \\
\xi_v(0, t) = \xi(l, t) = 0, & t \in [0, T].
\end{cases}
\]

**Proof.** For any \( h(t) \in R_0 \) and \( \delta \in [0, 1] \) let us set

\[
r_\delta := (1 - \delta) r + \delta h.
\]

Then we have

\[
J_\delta := J(r_\delta) = \frac{1}{2} \int_0^1 [\nu(x, 0; r_\delta) - g(x)]^2 \, dx + \frac{\alpha}{2} \int_0^T r_\delta^2(t) \, dt.
\]

Differentiation of \( J_\delta \) with respect to \( \delta \) at \( \delta = 0 \) gives the following inequality since \( r(t) \) is the solution of the optimal control problem (11), (12):

\[
\int_0^1 [r(x, 0) - g(x)] \frac{d\\nu(x, 0; r_\delta)}{d\delta} \, dx + \alpha \int_0^T r(t)[h(t) - r(t)] \, dt \geq 0.
\]

Let us denote by \( \nu_\delta = \nu_\delta(x, t; r_\delta) \) the corresponding solution of problem (7)–(10). Taking \( \nu_\delta = \frac{dp_\delta}{dt} \) then \( \nu_\delta \) is the solution of the following problem:

\[
\begin{cases}
\nu_{xx} = \nu_x + (h(t) - r(t)) f(x, t), & x \in (0, 1), \ t \in (0, T] \\
\nu(x, T) = \varphi(x) (h(T) - r(T)), & x \in [0, 1] \\
\nu_x(0, t) = \nu(l, t) = 0, & t \in [0, T].
\end{cases}
\]

(15)

Let us set \( \xi = \left. \nu_\delta \right|_{\delta=0} \) then \( \xi \) satisfies the following problem:

\[
\begin{cases}
\xi_v = \xi_{xx} + (h(t) - r(t)) f(x, t), & x \in (0, 1), \ t \in [0, T] \\
\xi(x, T) = 0, & x \in [0, 1] \\
\xi_v(0, t) = \xi(l, t) = 0, & t \in [0, T].
\end{cases}
\]

(16)

Taking into account (16) in (14) we obtain

\[
\int_0^1 [r(x, 0; r) - g(x)]\xi(x, 0) \, dx + \alpha \int_0^T r(t)[h(t) - r(t)] \, dt \geq 0.
\]

\[\square\]

**Lemma 3.** Let \( r(t) \) be the solution of the optimal control problem (11), (12). Then for any \( h(t) \in R_0 \) the following integral inequality holds:

\[
-\int_0^T \int_0^1 [h(t) - r(t)] f(x, t) \varphi(x, t) \, dx \, dt + \alpha \int_0^T r(t)[h(t) - r(t)] \, dt \geq 0
\]

(17)

where \( \varphi(x, t) \) is the solution of the adjoint problem given by

\[
\begin{cases}
-\varphi_t - \varphi_{xx} = 0, & x \in (0, 1), \ t \in (0, T] \\
\varphi(x, 0) = \nu(x, 0) - g(x), & x \in [0, 1] \\
\varphi_x(0, t) = \varphi(l, t) = 0, & t \in [0, T].
\end{cases}
\]

(18)

**Proof.** The unique solution \( \varphi(x, t) \) of (18) belongs to \( C^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, T]) \).

Multiplying each side of equation (18) by \( \xi(x, t) \) satisfies (16), integrating it by parts, we obtain
Finally, we obtain

\[ 0 = \int_0^T \int_0^1 \xi (-\varphi_t - \varphi_{xx}) dx dt = \int_0^T \int_0^1 \xi \varphi_t dx dt - \int_0^T \int_0^1 \xi \varphi_x dx dt - \int_0^T \int_0^1 \xi \varphi_{xx} dx dt + \int_0^T \xi \varphi_{x^2} dx dt + \int_0^T \xi \varphi_{x^4} dx dt \]

and using the conditions of (16), (18), the above integral identity implies:

\[ \int_0^T \int_0^1 [h(t) - r(t)] f(x, t) \varphi(x, t) dx dt + \int_0^T [v(x, 0; r) - g(x)] \xi(x, 0) dx = 0 \]

Combining (19) and (13), we obtain the integral inequality (17). \( \square \)

4. Uniqueness

**Theorem 4.** Suppose that \( g_1(x) \) and \( g_2(x) \) are two functions which satisfy (7)–(10). Let \( r_1(t) \) and \( r_2(t) \) be the solutions of the optimal control problem (11), (12) then we have the following estimate:

\[ \| r_1 - r_2 \|_{L^2(0, T)} \leq \frac{1}{\sqrt{2x}} \| g_1 - g_2 \|_{L^2(0, 1)} \].

**Proof.** Denote by \( \nu_i = \nu_i(x, t, \pi), i = 1, 2 \) the solutions of the problem (7)–(10) corresponding to \( r_1(t) \). Let \( \xi_1 = \xi(x, t, r_2 - r_1) \) and \( \xi_2 = \xi(x, t, r_1 - r_2) \) be two solutions of (16). Defining by \( w = \nu_1 - \nu_2 \) and \( \pi = \xi_1 + \xi_2, w(x, t) \) and \( \pi(x, t) \) are the solutions of the following problems, respectively:

\[
\begin{align*}
w_t &= w_{xx} + (r_1(t) - r_2(t)) f(x, t), \quad x \in (0, 1), \ t \in (0, T) \\
w(x, T) &= 0, \quad x \in [0, 1] \\
w_x(0, t) &= w(l, t) = 0, \quad t \in [0, T],
\end{align*}
\]

\[
\begin{align*}
\pi_t &= \pi_{xx}, \quad x \in (0, 1), \ t \in (0, T) \\
\pi(x, T) &= 0, \quad x \in [0, 1] \\
\pi_x(0, t) &= \pi(l, t) = 0, \quad t \in [0, T],
\end{align*}
\]

Note that we have \( w = -\xi_1 \) and \( w = \xi_2 \) so we conclude \( \pi(x, t) = 0 \). Now, taking \( h = r_{i-1} \) and \( r = r_i, \ i = 1, 2 \) into (13), respectively, there holds

\[ \int_0^1 [v_1(x, 0) - g_1(x)] \xi_1(x, 0) dx + \alpha \int_0^T r_1(t) |r_2(t) - r_1(t)| dt \geq 0. \]  

(22)

\[ \int_0^1 [v_2(x, 0) - g_2(x)] \xi_2(x, 0) dx + \alpha \int_0^T r_2(t) |r_1(t) - r_2(t)| dt \geq 0. \]  

(23)

Combining (22) and (23), we obtain

\[ \alpha \int_0^T [r_1(t) - r_2(t)]^2 dt \leq \int_0^1 [v_1(x, 0) - g_1(x)] \xi_1(x, 0) dx + \int_0^1 [v_2(x, 0) - g_2(x)] \xi_2(x, 0) dx \\
= \int_0^1 [v_1(x, 0) - g_1(x)] \xi_1(x, 0) dx - \int_0^1 [v_2(x, 0) - g_2(x)] \xi_2(x, 0) dx + \int_0^1 [v_2(x, 0) - g_2(x)] \xi_1(x, 0) dx \\
+ \int_0^1 [v_2(x, 0) - g_2(x)] \xi_2(x, 0) dx \\
= \int_0^1 [w(x, 0) - (g_1(x) - g_2(x))] \xi_1(x, 0) dx \\
\]

then let us use \( w(x, t) = -\xi_1(x, t) \) and Young’s inequality in the above inequality:

\[ \alpha \int_0^T [r_1(t) - r_2(t)]^2 dt \leq - \int_0^1 \xi_1^2(x, 0) dx + \frac{1}{2} \int_0^1 \xi_1^2(x, 0) dx + \frac{1}{2} \int_0^1 \xi_1^2(x, 0) dx \\
= \frac{1}{2} \int_0^1 \xi_1^2(x, 0) dx - \frac{1}{2} \int_0^1 \xi_1^2(x, 0) dx \\
\leq \frac{1}{2} \int_0^1 (g_1(x) - g_2(x))^2 dx. \]

Finally, we obtain

\[ \| r_1 - r_2 \|_{L^2(0, T)} \leq \frac{1}{\sqrt{2x}} \| g_1 - g_2 \|_{L^2(0, 1)}. \]  

\( \square \)
References


