Determination of Eigenvalues of Closed Lossless Waveguides Using the Least Squares Optimization Technique

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Abstract—It is known that in lossless and closed guides filled with gyroelectric or gyromagnetic media, Maxwell’s equations are transformed into an infinite linear algebraic equation system by application of the Galerkin version of Moment method. Propagation constant of the problem whose exact solution is not known is found as the square root of the eigenvalue of the coefficient matrix of this infinite linear algebraic equation system. However the quantities determined from the Moment method do not have much physical meaning by themselves. This is why it is desired to express the propagation constant as a function of frequency. Therefore with the help of algebraic function theory we attempt to express eigenvalues of the coefficient matrix, i.e., the squares of the propagation constants, as a series expansion. By this technique Laurent and Puiseux series expansions in the neighborhood of singular points are obtained for these functions. The series expansions which are obtained by the algebraic function approximation in the eigenvalue problems of closed, lossless, uniform waveguides, bring about a function theoretic insight in order to investigate the properties of the propagation constant functions. In previous work reported in the literature there exists a derivation of these expansions accomplished by using differentiation of the characteristic equation of above coefficient matrix, considering it as an implicit function of the propagation constant function and the frequency. In this work computation of the necessary expansion coefficients of the Laurent and Puiseux series, is achieved by the least squares technique (LST) which is a curve fitting method. In this way we attempt to find a simple solution to the problem of computation of these coefficients, which dates back to Sir Isaac Newton and in general which is rather tedious.

1. INTRODUCTION

Maxwell’s partial differential equations and boundary conditions for closed waveguides loaded with heterogeneous and/or anisotropic medium, are transformed into a coupled system of an infinite number of ordinary differential equations (ODE) called transmission line equations [1]. In uniform waveguides these equations are transformed into a linear algebraic equation system because then z dependence has the form $e^{-\gamma z}$, where $\gamma$ is the propagation constant. When the filling medium is gyrotropic this equation system takes the particular form:

$$\gamma(p) \begin{bmatrix} v(p) \\ i(p) \end{bmatrix} = \begin{bmatrix} 0 & Z(p) \\ Y(p) & 0 \end{bmatrix} \begin{bmatrix} v(p) \\ i(p) \end{bmatrix} \quad (1)$$

In (1), $Z(p)$ and $Y(p)$ square matrices with infinite dimensions are truncated to be $n \times n$ as an approximation [2–4]. Linear algebraic equation system in (1) is in particular an eigenvalue problem. When $Z(p)$ and $Y(p)$ are truncated, call the eigenvalue of the matrix product $Z(p)Y(p)$ or $Y(p)Z(p)$, which two matrices share the same eigenvalues, as $\gamma_1^2(p)$. $v(p)$ and $i(p)$ are then eigenvectors of these matrix products $Z(p)Y(p)$ and $Y(p)Z(p)$, respectively. When the polynomial consisting of the characteristic equation of these matrices is set equal to zero and is multiplied by the common denominator of its terms, the characteristic equation of $Z(p)Y(p)$ (or $Y(p)Z(p)$) becomes as in (2).

$$G(\gamma_1^2, p) = a_0(p) \gamma_1^{2n}(p) + a_1(p) \gamma_1^{2(n-1)}(p) + \ldots + a_{n-1}(p) \gamma_1^2(p) + a_n(p) = 0 \quad (2)$$

In fact, the method implemented to obtain (1) is the Galerkin version of Moment method. This is because expansion functions have been taken equal to test functions in the Moment method. However the quantities determined from the Moment method do not have much physical meaning by themselves. Because they are merely data points for propagation constants versus frequency. That is why we expand the function (the roots) into Puiseux and Laurent series. Puiseux series is a power series with fractional powers. The information obtained from these series gives us a physical insight and this is why it is desired to expand the roots of the algebraic equation into series [2].
The aim of this work is to compute by the method of optimization the expansion coefficients in Laurent and Puiseux series of squares of propagation constants obtained by the Moment method through use of elementary algebraic function theory. These coefficients were determined using approximate expressions based on differentiations of (2) considering it as an implicit function in [2]. In [2] only two expansion coefficients were found by the described technique. If additional coefficients are required, operational load will increase sharply. The difficulty in determining the coefficients by Newton’s solution with polygons exists for our problem. Because the explicit analytic coefficients of the characteristic equation are impossible to obtain at least in our case, determining the coefficients by Newton polygons is impossible.

The impossibility of analytic determination of these coefficients explicitly necessarily leads us to resort to use of optimization methods as an alternative. To compute the expansion coefficients we shall use the least squares method which is an optimization method. Although Newton’s solution with polygons is suitable for programming, the least squares technique is introduced in this paper for our problem as a simple alternative due to above reasons.

2. MODELING OF DISPERSION EQUATION USING THE LST

In (2), $G(\gamma_1^2, p) = 0$ equality has $n$ different roots for each $p$. An exception occurs only when a) $a_0(p_0) = 0$, b) $G(\gamma_1^2, p)$ has multiple roots. The exception under a) and b) are only for a finite number of special values of $p$ which are the critical points. These critical points constitute only algebraic singularities for $\gamma_1^2(p)$ (please see [3]). Thus let solution of approximate propagation constant of our problem in the neighborhood of a singularity $p_0 = j\omega_0$ be $\gamma_1^2(p)$ in (3).

$$\gamma_1^2(p_i) = \sum_{n=n_2}^{\infty} C_n \rho_i^n \quad (3)$$

Here, if $n_2 < 0$ and $\rho_i = (p_i - p_0)$. Equation (3) is a Laurent series, if $n_2 = 0$ and $\rho_i = (p_i - p_0)^{1/q}$, Equation (3) is a Puiseux series. It has been established that pole branch points are ruled out for all $n_2$. These critical points constitute only algebraic singularities for $\gamma_1^2(p)$ (please see [3]). Thus let solution of approximate propagation constant of our problem in the neighborhood of a singularity $p_0 = j\omega_0$ be $\gamma_1^2(p)$ in (3).

$$\gamma_1^2(p_i) = \sum_{n=n_2}^{N} \tilde{C}_n \rho_i^n \quad (4)$$

Here $\tilde{C}_n$ are the approximate coefficients to take places of $C_n$ in (3) and our problem is to determine the $\tilde{C}_n$ optimally to best approximate (3) by (4). At point $p_i$ when the error between approximate function $\gamma_1^2$ and exact dispersion equation solution $\gamma^2(p_i)$ is shown by $s_i$, it can be defined as follows.

$$s_i = \gamma^2(p_i) - \gamma_1^2(p_i) = \gamma^2(p_i) - \sum_{n=n_2}^{N} \tilde{C}_n \rho_i^n \quad (5)$$

The square of propagation constant is a function of a complex variable. If the error is required to be a real and positive value, the error for $M$ frequency points can be defined as $S$ in (6).

$$S = \sum_{i=1}^{M} \left| \gamma^2(p_i) - \gamma_1^2(p_i) \right|^2 = \sum_{i=1}^{M} \left[ \gamma^2(p_i) - \gamma_1^2(p_i) \right] \times \left[ \gamma^2(p_i) - \gamma_1^2(p_i) \right]^* \quad (6)$$

According to the LST, derivative of the error function with respect to expansion coefficients of approximate function of (4) must be zero, in order for the full error to be minimum.

Approximate function $\gamma_1^2(p)$ is a complex function, so the expansion coefficient $\tilde{C}_n$ is a function of real variables $\overline{C}_n$ and $\overline{C}_n$ where $\tilde{C}_n = \overline{C}_n + j\overline{C}_n$. When coefficients of (4) are defined as $\tilde{C}_n = \overline{C}_n + j\overline{C}_n$ and substituted for $\gamma_1^2$ in (6) rearranged, Equation (7) is obtained.

$$S = \sum_{i=1}^{M} \left[ \gamma^2(p_i) - \sum_{n=n_2}^{N} \left\{ \overline{C}_n + j\overline{C}_n \right\} \rho_i^n \right] \times \left[ \gamma^2(p_i) - \sum_{n=n_2}^{N} \left\{ \overline{C}_n + j\overline{C}_n \right\} \rho_i^n \right]^* = 0 \quad (7)$$
In order to devise a minimization algorithm for determination of the optimum coefficients based on the LST, it is necessary to evaluate the partial derivatives of $S$ with respect to $C_i$ and $\overline{C}_i$ [6] and equate them to zero. Then denoting the complex conjugate by ($*$), we obtain the following equations for $m$th coefficient where $n_2 < m < N$:

$$\frac{\partial S}{\partial C_m} = \sum_{i=1}^{M} \rho_i^m \left[ \gamma^2(p_i) - \sum_{n=n_2}^{N} \left\{ C_n + j\overline{C}_n \right\} \rho_i^n \right] + \sum_{i=1}^{M} \rho_i^{m*} \left[ \gamma^2(p_i) - \sum_{n=n_2}^{N} \left\{ \overline{C}_n + jC_n \right\} \rho_i^n \right] = 0 \quad (8)$$

$$\frac{\partial S}{\partial \overline{C}_m} = \sum_{i=1}^{M} j\rho_i^m \left[ \gamma^2(p_i) - \sum_{n=n_2}^{N} \left\{ C_n + j\overline{C}_n \right\} \rho_i^n \right] + \sum_{i=1}^{M} j\rho_i^{m*} \left[ \gamma^2(p_i) - \sum_{n=n_2}^{N} \left\{ \overline{C}_n + jC_n \right\} \rho_i^n \right] = 0 \quad (9)$$

In the following two examples, we have the complex frequency $p_i = j\omega_i$ in all of our formulas.

### 3. EXAMPLE 1: DIELECTRIC ROD LOADED CYLINDRICAL GUIDE

First we consider a dielectric rod loaded cylindrical guide (inset of Figure 1). Radius of the guide is 0.25" whereas ratio of radii of rod and guide is 0.67. Relative dielectric permittivity of rod is 15. The normalized frequency $j\omega_0$ is a zero of the discriminant of (2) and this singularity is an algebraic branch point where the eigenvalue is finite and multiple. Then in Equation (3), $q = 2$ is the degree of cycle of functions with an algebraic branch point at $j\omega_0$ [2], in other words there are two multiple roots at this point.

We obtain the Puiseux series expansion in the neighborhood of the algebraic branch point for one root of Equation (2) using properties of algebraic Equation (2) [7].

$$\gamma_1^2(j\omega) = \gamma_0^2(j\omega_0) + A_1 \sqrt{j\omega - j\omega_0} + A_2 (j\omega - j\omega_0) + \ldots + A_n \left( \sqrt{j\omega - j\omega_0} \right)^n + \ldots \quad (10)$$

The propagation constant versus frequency has been recomputed with the help of the series expansion which is obtained by taking the first four optimum coefficients and is shown together with the exact solution in Figure 1. Optimization is built in the interval $0.94297 < V < 0.97298$ and the coefficients of (10) are computed by the LST in this interval. The approximate propagation constant has been recalculated in the interval $0.5094 < V < 1.19798$ using these optimum first four coefficients. Results obtained by exact dispersion equation and proposed method are illustrated in the normalized frequency range $0.5094 < V < 1.19798$ in Figure 1, where $V = \omega/\sqrt{\varepsilon_0\mu_0}r_1$ is the normalized frequency.

As it is seen, between the solution of exact dispersion relation and the LST a very close agreement has been achieved on a very wide frequency band. The coefficients found by help of this optimization method are as in Table 1(a).

The close agreement between the solution of optimization method and [2] for the coefficients $A_1$ and $A_2$ confirms the validity of optimization method used (see Table 1(a)). Because of the

![Figure 1: Comparison of exact solution and present method for propagation constant. Inset: Guide cross-section for Example 1.](image-url)
Table 1: (a) Expansion coefficients of Puiseux series for Example 1 by the LST and method in [2], (b) expansion coefficients of Laurent series for Example 2 by the LST and method in [2].

<table>
<thead>
<tr>
<th>Coefficients of Puiseux series</th>
<th>Computed by the LST</th>
<th>Computed in [2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>5.2687($1 - j$)</td>
<td>5.259648($1 - j$)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$0.4775 \times 10^{-8} + j14.9630$</td>
<td>$j19.3257$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$-7.7460 - j7.7460$</td>
<td>-</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$6.4291 - j0.1141 \times 10^{-6}$</td>
<td>-</td>
</tr>
</tbody>
</table>

(a)

<table>
<thead>
<tr>
<th>Coefficients of Laurent series</th>
<th>Computed by the LST</th>
<th>Computed in [2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{-1}$</td>
<td>$0.4597 \times 10^{-7} + j1.7258 \times 10^5$</td>
<td>$j1.7809 \times 10^4$</td>
</tr>
<tr>
<td>$C_0$</td>
<td>$1.4580 \times 10^8 - j0.7217 \times 10^{-2}$</td>
<td>$1.7234 \times 10^8$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$0.3004 \times 10^{-2} + j7.8581 \times 10^4$</td>
<td>-</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$-53.9799 + j0.1880 \times 10^{-5}$</td>
<td>-</td>
</tr>
</tbody>
</table>

(b)

flexibility of algorithm used, in the LST the number of coefficients can be increased at will, in order to better approximate the exact dispersion equation. The coefficients can be found very efficiently using the derived algorithm.

4. EXAMPLE 2: FERRITE TUBE LOADED CYLINDRICAL GUIDE

Next, we consider a structure consisting of a ferrite tube filled cylindrical waveguide (inset of Figure 2(a)). The permeability tensor for the ferrite medium magnetized axially is given in (11) [8].

$$\hat{\mu} = \mu_0 \begin{bmatrix} \mu_r & -j\kappa & 0 \\ j\kappa & \mu_r & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\mu_r = 1 - \frac{\omega_0 \omega_m}{\omega^2 - \omega_0^2}$ and $\kappa = \frac{\omega \omega_m}{\omega^2 - \omega_0^2}$ (11)

The parameter values of the structure are as follows [8]. $r_1 = 8\, mm$, $r_0/r_1 = 5/8$, $\varepsilon_1 = 12.9\varepsilon_0$, resonance frequency $f_0 = \omega_0/(2\pi) = 6.998206\, GHz$, $f_m = \omega_m/(2\pi) = 0.6683\, GHz$. For this structure $a_0(j\omega_0)$ in (2), vanishes at resonance frequency $\omega_0$ and at least one root of (2) attains

![Figure 2: (a) Phase coefficient computed by Moment method, by the LST method and the method in [2] Inset: Guide cross-section for Example 2, (b) attenuation constant of the evanescent mode obtained by Moment, the LST method and the method in [2].](Image)
an infinite value at this point. This means that $\omega_0$ is a singularity point. Actually $Z(p)$ and $Y(p)$ matrices constructed for Moment method have a pole at $j\omega_0$. On the other hand a pole branch point is not possible in the structures of this type \[2\]. Therefore Laurent expansion is obtained as (12) in the neighborhood of pole point $j\omega_0$ at which $a_0(j\omega_0) = 0$.

$$
\gamma_1^2(p) = \frac{C_{-m}}{(p - j\omega_0)^m} + \ldots + \frac{C_{-1}}{(p - j\omega_0)} + C_0 + C_1(p - j\omega_0) + \ldots \quad (12)
$$

Number of negative power terms or the order of pole in (12) has been found as $m = 1$ \[9\]. For the proposed method result closest to the exact dispersion relation has been found when the first four coefficients of Laurent series expansions are taken into consideration. These coefficients are given in Table 1(b). The results obtained for the phase coefficient by the help of Moment method, proposed method and method in \[2\] are indicated in Figure 2 for the first mode of right hand circularly polarized wave. The close agreement between the solution of the LST and \[2\] (see Table 1(b)) for the first two coefficients confirms the validity of used method. The values found for the coefficients are also used to compute the propagation constant above the ferrite resonance frequency as illustrated in Figure 2(b).

REFERENCES