Exact and approximate solutions of second order including function delay differential equations with variable coefficients

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Abstract

By means of the method of steps and the method of power series, the exact solution of the initial value problem for coupled second order delay differential equations with variable coefficients is constructed. Then, in a bounded domain, a finite analytic solution with error bounds is provided.

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1. Introduction

In many fields of the contemporary science and technology systems with delaying links are often met and the dynamical processes in these are described by systems of delay differential equations [2–4]. The delay appears in complicated systems with logical and computing devices, where certain time for information processing is needed.

The theory of linear delay differential equations has been developed in the fundamental monographs [2–4,6]. Analytic solutions of some linear systems of delay differential equations have been investigated in [1,5,7].

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In this paper we consider initial value problems for systems of second order delay differential equations

\[
\begin{align*}
  x''(t) + a(t)x'(t) + b(t)x(t) + b_1(t)x(t - \Delta(t)) &= f(t), & 0 \leq t < a, \\
  x(t - \Delta(t)) &= g(t - \Delta(t)), & t - \Delta(t) < 0, \\
  x(0) &= g(0), & x'(0 + 0) = g'(0),
\end{align*}
\]

(1.1)

where \(a(t)\) and \(b(t)\) are analytic complex valued functions in \(|t| < a\), \(b_1(t)\) is a complex valued continuous function, the unknown \(x(t)\) as well as \(f(t)\) and \(g(t)\) are complex valued functions, with \(f(t)\) continuous in \(0 \leq t < a\) and \(g(t)\) is a continuously differentiable function in an \(E_0\). Deviation \(\Delta(t)\) defines an initial set \(E_0\), consisting of the point \(0\) and those values \(t - \Delta(t)\) for which \(t - \Delta(t) < 0\) if \(t \geq 0\). Here \(\Delta(t) \geq d > 0\) delay function is continuous and bounded in the interval \([0, a)\). Furthermore, let \(\sup_{t \in [0,a)} \Delta(t) = m\) and inequality \(0 < d \leq m < a < +\infty\). The function \(t - \Delta(t)\) be strictly monotone increasing on each half-open interval \([0, a)\).

The aim of this paper is twofold. First of all we construct a series solution of problem (1.1) by means of a method of power series and the method of steps, but dealing directly with (1.1). Secondly, we truncate the series solution and provide error bounds for the continuous finite approximate solution when \(t \in [\gamma_{n-1}(0), \gamma_n(0)]\) and \(n\) is a positive integer. For constant coefficient case, systems of second order delay differential equations have been recently studied in [1,5,7] avoiding the transformation of the problem into an equivalent extended first order system.

This paper is organized as follows. In Section 2 we construct a series solution of problem (1.1). Error analysis of the finite truncated series in terms of the data, for a given interval \([\gamma_{n-1}(0), \gamma_n(0)] \subset [0, a)\), is studied in Section 3.

### 2. A series solution of the problem

We begin this section considering the homogeneous differential system

\[
\begin{align*}
  x''(t) + a(t)x'(t) + b(t)x(t) &= 0, \\
\end{align*}
\]

(2.1)

where \(a(t)\), \(b(t)\) are complex valued analytic functions in \(|t| < a\) and

\[
\begin{align*}
  a(t) &= \sum_{i=0}^{\infty} a_i t^i, & b(t) &= \sum_{i=0}^{\infty} b_i t^i, & |t| < a.
\end{align*}
\]

(2.2)

The coefficients \(a_i, b_i\) satisfy the Cauchy inequalities

\[
|a_i| \ell^i \leq L, \quad |b_i| \ell^i \leq L, \quad 0 < \ell < a, \quad v \in \mathbb{N} \cup \{0\}
\]

(2.3)
for some positive constant \( L \). Let us look for solutions of (2.1) of the form
\[
x(t) = \sum_{v=0}^{\infty} c_v t^v,
\]
where the coefficients \( c_v \) are to be determined. We get the recurrent equations
\[
(v + 2)(v + 1)c_{v+2} = -\sum_{j=0}^{v} ((j + 1)a_{v-j}c_{j+1} + b_{v-j}c_j), \quad v \geq 0, \tag{2.4}
\]
where \( c_0, c_1 \) are arbitrary complex numbers. Making use of (2.3), we can show that for any pair of complex numbers \( c_0 \) and \( c_1 \) the series \( x(t) = \sum_{v=0}^{\infty} c_v t^v \) with coefficient \( c_v \) defined by (2.4) is absolutely convergent in \( |t| < a \). Let us denote by \( x_1(t) \) the solution of (2.1) constructed by the above procedure with \( c_0 = 1, c_1 = 0 \), and let \( x_2(t) \) be defined in the same way with \( c_0 = 0, c_1 = 1 \). Then the pair \( \{x_1, x_2\} \) is a fundamental system of solutions of (2.1) in \( |t| < a \), and the set of all complex solutions of (2.1) in \( |t| < a \) is given by
\[
\Phi(t) = x(0)x_1(t) + x'(0)x_2(t). \tag{2.5}
\]

Now let \( h(t) \) be a continuous complex valued function in \( |t| < a \), and let us consider the nonhomogeneous equation
\[
x''(t) + a(t)x'(t) + b(t)x(t) = h(t), \quad |t| < a. \tag{2.6}
\]
The matrix
\[
W(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} \tag{2.7}
\]
is invertible in \( |t| < a \), and its inverse matrix is
\[
V(t) = W^{-1}(t) = \exp \left( \int_{0}^{t} A(s) \, ds \right) \begin{pmatrix} x_2'(t) & -x_2(t) \\ -x_1'(t) & x_1(t) \end{pmatrix} = \begin{pmatrix} v_{11}(t) & v_{12}(t) \\ v_{21}(t) & v_{22}(t) \end{pmatrix}. \tag{2.8}
\]

By the variation of parameters method, the general solution of the linear equation (2.6) is given by
\[
x(t) = \Phi(t) + \int_{0}^{t} (x_1(t)v_{12}(\tau) + x_2(t)v_{22}(\tau)) h(\tau) \, d\tau.
\]

We introduce the integral operator \( U \) defined by
\[
(Uh)(t) = \int_{0}^{t} k(t, \tau) h(\tau) \, d\tau, \quad 0 \leq t < a, \tag{2.9}
\]
where the kernel \( k(t, \tau) \) is given by
\[
k(t, \tau) = x_1(t)v_{12}(\tau) + x_2(t)v_{22}(\tau). \tag{2.10}
\]
It is clear that $k(t, t) = 0$ and that the operator $U$ is linear, i.e.

$$U(\alpha h + \beta m) = \alpha Uh + \beta Um.$$  

Here and below, when possible, we will drop the dependence on $t$ for the sake of brevity.

Now we can state the result obtained as follows.

**Theorem 1.** Consider equation (2.6) where $a(t), b(t)$ are analytic complex valued functions with power expansions (2.2), and let $h(t)$ be a continuous function. Let \( \{x_1(t), x_2(t)\} \) be the pair of analytic solutions of (2.1) satisfying $x_1(0) = 1$, $x_1'(0) = 0$, $x_2(0) = 0$, $x_2'(0) = 1$. Then the unique solution $x(t)$ of (2.6) satisfying the initial conditions $x(0) = c_1$, $x'(0) = c_2$ is given by

$$x = c_1 x_1 + c_2 x_2 + Uh, \quad (2.11)$$

where the operator $U$ is defined by (2.9).

Now, for each $t \in [0, a)$ a function should be defined as following:

$$\gamma(t) = \sup\{\tau \in [0, a) : \tau - A(\tau) < t\}.$$  

Because of $t - A(t)$ being a strictly monotone increasing $\gamma(t) > t$ is obvious for $\forall t \in [0, a)$ and let

$$0 = \gamma_0(0), \gamma_1(0) = \gamma(0), \ldots, \gamma_n(0) = \gamma(\gamma_{n-1}(0)), \ldots$$

If, for some $n$, $\gamma_n(0) = a$, then we define $\gamma_{n+k}(0) = a$, $k = 1, 2, \ldots$ The process of successive integrations on the intervals $[\gamma_{n-1}(0), \gamma_n(0)]$, $n = 1, 2, \ldots$ can be extended to the whole interval $[0, a)$. Among these intervals

$$\inf_n (\gamma_n(0) - \gamma_{n-1}(0)) > 0$$

is obvious.

Let us consider the delay differential equation (1.1) written in the form

$$x''(t) + a(t)x'(t) + b(t)x(t) = f(t) - b_1(t)x(t - A(t)), \quad t \geq 0,$$

$$x(t - A(t)) = g(t - A(t)), \quad t - A(t) < 0,$$  

(2.12)

where $a(t), b(t)$ are analytic complex valued functions, and note that for $t \in [0, \gamma(0)]$ the right-hand side of (2.12) is a known continuous function. Then by Theorem 1 the solution of (2.12) can be written in the form

$$x = \Phi + U(f - b_1x_{-A}) = \Phi + Uh - U(b_1x_{-A}), \quad (2.13)$$

where by definition

$$x_{-A}(\tau) = x(\tau - A(\tau))$$

and

$$\Phi(t) = g(0)x_1(t) + g'(0)x_2(t), \quad 0 \leq t < a. \quad (2.14)$$
Note that (2.13) is a feedback expression that gives the solution of (1.1) in an interval of length $\Delta(t)$ in terms of the solution in the previous intervals of length $\Delta(t)$. In order to find a closed form expression for the solution of (1.1) in any interval $[\gamma_{n-1}(0), \gamma_n(0)] \subset [0,a)$, we introduce a recurrent sequence of integral operators. If $h$ is a continuously differentiable function in $E_0 \cup [0,a)$, we define

$$U_1(b_1 h) = U(b_1 h)$$  (2.15)

and for an integer $k > 1$

$$U_k(b_1 h) = U(b_1(U_{k-1}(b_1 h))_{-\Delta}).$$  (2.16)

Here, as above, $(U_{k-1}(b_1 h))_{-\Delta}(t) = (U_{k-1}(b_1 h))(t - \Delta(t))$. Now by induction on $n$ we prove that the solution of (1.1) can be written in the compact form

$$x(t) = g(t), \quad t \in E_0$$  (2.17)

and for $t \in [\gamma_{n-1}(0), \gamma_n(0)]$

$$x = \Phi + Uf + \sum_{k=1}^{n} (-1)^k U_k(b_1(Uf)_{-\Delta}) + \sum_{k=1}^{n} (-1)^k U_k(b_1 \Phi_{-\Delta})$$

$$+ (-1)^{n+1} U_{n+1}(b_1 g_{-\Delta}).$$  (2.18)

This proves the following result.

**Theorem 2.** Let us consider problem (1.1) under the hypothesis of Theorem 1. Let $\{U_k\}_{k \geq 1}$ be the sequence of operators defined by (2.15) and (2.16), where $b_1(t)$ is a complex valued continuous function and let $k(t, \tau)$ be defined by (2.10). If $\Phi(t)$ is defined by (2.14), then the exact solution of (1.1) in the interval $[\gamma_{n-1}(0), \gamma_n(0)] \subset [0,a)$ for $n \geq 0$ is given by (2.18).

**Remark.** It is easy to show that the integral operators $U_k$ defined by (2.15) and (2.16) can be written in terms of the data in the form

$$(U_k(b_1 h))(t) = \int_0^t \int_0^{t_n - \Delta(t_n)} \cdots \int_0^{t_{p+1} - \Delta(t_{p+1})} [k(t, t_n) b_1(t_n) k(t_n - \Delta(t_n), t_n-1) \cdots$$

$$\cdots b_1(t_{p+1}) k(t_{p+1} - \Delta(t_{p+1}), t_p) b_1(t_p) h(t_p)] dt_p dt_{p+1} \cdots dt_n,$$  (2.19)

where $p = n - k + 1$.

From a computational point of view, the solution provided by Theorem 2 has the drawback that the expression of $k(t, \tau)$ and $\Phi(t)$ is given in terms of infinite series involving $x_1(t), x_2(t)$ and $e^{a(t)}$ defined by Theorem 1. In the sequel we construct finite approximate solutions of (1.1) obtained by truncation of the quoted infinite series.
3. Finite analytic approximate solutions and error bounds

Let

\[ x_1(t) = \sum_{i=0}^{\infty} c_i t^i, \quad x_2(t) = \sum_{i=0}^{\infty} d_i t^i, \quad |t| < a, \]  

(3.1)

be the pair of analytic series solutions of (2.1) given by Theorem 1. For any positive integer \( i \) let us define the truncated series

\[ x_i^1(t) = \sum_{i=0}^{i} c_i t^i, \quad x_i^2(t) = \sum_{i=0}^{i} d_i t^i, \]  

(3.2)

\[ \Phi'(t) = g(0)x_i^1(t) + g'(0)x_i^2(t). \]  

(3.3)

In this part, we will find error bound between exact solution \( x(t) \) and approximate solution \( x'(t) \) in intervals \([\gamma_{n-1}(0), \gamma_n(0)]\), \( n \in \mathbb{N} \). Here, for a matrix \( P \) in \( C^{2 \times 2} \) we denote by \( \|P\| \) the \( \ell_1 \) norm of \( P \).

In order to evaluate the error due to this truncation on some fixed interval \([\gamma_{n-1}(0), \gamma_n(0)] \subset [0, a] \), We shall need some estimates. First choose \( \theta > 0 \) so that \( \gamma_n(0)(\theta + 1) < a \). Then for a fixed nonnegative integer \( q \leq n \) we denote

\[ R = \gamma_q(0)(\theta + 1). \]  

(3.4)

In this sense, functions \( x_1(t), x_2(t), x'_1(t) \) and \( x'_2(t) \) are bounded in interval \( t \in [0, R] \) and in this interval there are positive numbers \( k_1, k_2, k_3, k_4 \) in order to have

\[ |x_1(t)| \leq k_1, \quad |x_2(t)| \leq k_2, \quad |x'_1(t)| \leq k_3, \quad |x'_2(t)| \leq k_4. \]  

(3.5)

We will have

\[ \left| \exp \left( \int_0^t a(s) \, ds \right) \right| \leq m_R, \quad t \in [0, R]. \]  

(3.6)

We may find number \( S_q \) to get

\[ \|W(t)\| = |x_1(t)| + |x_2(t)| + |x'_1(t)| + |x'_2(t)| \leq S_q, \]  

(3.7)

\[ \|V(t)\| = \left| \exp \left( \int_0^t a(s) \, ds \right) \right| \left( |x'_2| + | - x_2| + | - x'_1| + |x_1| \right) \leq m_R(k_1 + k_2 + k_3 + k_4) \leq S_q \]  

(3.8)

by applying (3.5) and (3.6)

\[ \max\{|x_1(t)|, |x'_1(t)|, |x_2(t)|, |x'_2(t)|, |v_{12}(t)|, |v_{22}(t)|\} \leq S_q \]  

(3.9)

are obvious in interval \( t \in [0, R] \).
From the Cauchy inequalities it follows that:

$$|c_v| \leq S_q R^{-e}, \quad |d_v| \leq S_q R^{-e},$$

where \( R \) was defined by (3.4). Then for \( 0 \leq t \leq \gamma_q(0) \) we have

$$|x_1(t) - x'_1(t)| \leq \sum_{v=i+1}^{\infty} |c_v| |t|^v \leq S_q \sum_{v=i+1}^{\infty} \left( \frac{|t|}{R} \right)^v$$

$$\leq S_q \sum_{v=i+1}^{\infty} \left( \frac{\gamma_q(0)}{\gamma_q(0)(\theta + 1)} \right)^v$$

$$= S_q \sum_{v=i+1}^{\infty} (1 + \theta)^{-v} = \frac{S_q}{\theta(1 + \theta)^j} = S_q E_i,$$  \( 3.10 \)

and, analogously,

$$|x_2(t) - x'_2(t)| \leq S_q E_i.$$  \( 3.11 \)

Obviously,

$$\lim_{i \to \infty} E_i = 0.$$  \( 3.12 \)

Then we deduce

$$|x'_j(t)| \leq |x_j(t) - x'_j(t)| + |x_j(t)| \leq S_q(1 + E_i), \quad j = 1, 2.$$  \( 3.13 \)

$$|\Phi(t) - \Phi'(t)| = |g(0)x_1(t) + g'(0)x_2(t) - g(0)x'_1(t) - g'(0)x'_2(t)|$$

$$\leq (|g(0)| + |g'(0)|)S_q E_i,$$  \( 3.13 \)

$$|\Phi(t)| = |g(0)x_1(t) + g'(0)x_2(t)| \leq (|g(0)| + |g'(0)|)S_q,$$

$$|\Phi'(t)| = |g(0)x'_1(t) + g'(0)x'_2(t)| \leq (|g(0)| + |g'(0)|)S_q(1 + E_i).$$  \( 3.13 \)

Further on, for \( 0 \leq t \leq \gamma_q(0) \) and \( j = 1, 2 \) we obtain

$$|x'_j(t) - (x'_j)'(t)| \leq S_q \sum_{v=i+1}^{\infty} v|t|^{v-1} R^{-v} \leq \frac{S_q}{R} \sum_{v=i+1}^{\infty} v(1 + \theta)^{1-v} \leq \frac{S_q D_q}{q + 1}.$$  \( 3.14 \)

Obviously,

$$\lim_{i \to \infty} D_i = 0.$$  \( 3.15 \)

Let us denote

$$W_i(t) = \begin{pmatrix} x'_1(t) & x'_2(t) \\ (x'_1)'(t) & (x'_2)'(t) \end{pmatrix}$$

and represent it in the form
\[ W_i(t) = W(t) + (W(t) - W(t)) = W(t)[I + W^{-1}(t)(W_i(t) - W(t))], \]
\[ W_i(t) - W(t) = \begin{pmatrix} x_1'(t) - x_1(t) & x_2'(t) - x_2(t) \\ (x_1')'(t) - x_1'(t) & (x_2')'(t) - x_2'(t) \end{pmatrix}. \]

(3.16)

We have by virtue of (3.7), (3.8), (3.10), (3.11), and (3.14) that
\[
\|W_i(t) - W(t)\| \leq 2S_q\left( E_i + \frac{D_i}{q + 1} \right),
\]
\[
\|W^{-1}(t)(W_i(t) - W(t))\| \leq \|V(t)\| \cdot \|W_i(t) - W(t)\| \leq 2S_q^2\left( E_i + \frac{D_i}{q + 1} \right). \]

(3.17)

In view of (3.12) and (3.15) we can choose \( i_0 \) so large that for all integers \( i \geq i_0 \)
\[
\|W^{-1}(t)(W_i(t) - W(t))\| \leq 2S_q^2\left( E_i + \frac{D_i}{q + 1} \right) < 1. \]

(3.18)

Then, using the representation (3.16), we find that the matrix \( W_i(t) \) is invertible
for \( i \geq i_0 \) and
\[
\|(I + W^{-1}(t)(W_i(t) - W(t)))^{-1}\| \leq 1 + \frac{1}{1 - 2S_q^2(E_i + (D_i/q + 1))} = M_{iq},
\]
\[
W_i^{-1}(t) = (I + W^{-1}(t)(W_i(t) - W(t)))^{-1}W^{-1}(t),
\]
\[
\|W_i^{-1}(t)\| \leq \|(I + W^{-1}(t)(W_i(t) - W(t)))^{-1}\| \cdot \|W^{-1}(t)\| \leq S_qM_{iq}
\]

(3.19)

for \( 0 \leq t \leq \gamma_q(0) \).

Making use of the equality
\[
W^{-1}(t) - W_i^{-1}(t) = W_i^{-1}(t)(W_i(t) - W(t))W^{-1}(t)
\]
and of estimates (3.8), (3.17), and (3.19), we obtain
\[
\|W^{-1}(t) - W_i^{-1}(t)\| \leq \|W_i^{-1}(t)\| \cdot \|(W_i(t) - W(t))\| \cdot \|W^{-1}(t)\|
\]
\[
\leq 2S_q^3\left( E_i + \frac{D_i}{q + 1} \right)M_{iq} \quad \text{for} \ 0 \leq t \leq \gamma_q(0). \]

(3.20)

Now we define the approximate kernel
\[
k_i(t, \tau) = x_1'(t)v_{12}(\tau) + x_2'(t)v_{22}(\tau).
\]
From (3.9), (3.10), (3.11), and (3.19) it follows that:

\[
|k_i(t, \tau) - k(t, \tau)| = |x'_1(t)(v'_{12}(\tau) - v_{12}(\tau)) - v_{12}(\tau)(x'_1(t) - x'_1(t)) + x'_2(t)(v'_{22}(\tau) - v_{22}(\tau)) - v_{22}(\tau)(x'_2(t) - x'_2(t))| \\
\leq |x'_1(t)| \cdot |v'_{12}(\tau) - v_{12}(\tau)| + |v_{12}(\tau)| \cdot |x'_1(t) - x'_1(t)| + |x'_2(t)| \cdot |v'_{22}(\tau) - v_{22}(\tau)| + |v_{22}(\tau)| \cdot |x'_2(t) - x'_2(t)| \\
\leq 2S_q(1 + E_i)\left(2S_q^3 \left( E_i + \frac{D_i}{q + 1} \right) M_{iq} \right) + 2S_q^2 E_i \\
\equiv 2S_q^2 \gamma_q \rightarrow 0, \quad i \rightarrow \infty, \quad (3.21)
\]

\[
|k(t, \tau)| \leq |x'_1(t)| \cdot |v_{12}(\tau)| + |x'_2(t)| \cdot |v_{22}(\tau)| \leq 2S_q^2, \quad (3.22)
\]

Now for \( i \geq i_0, 0 \leq t < a \) we define the operators

\[
(U^ih)(t) = \int_0^t k_i(t, \tau)h(\tau) \, d\tau; \\
U_i^1(b_i h) = U^i(b_i h), \quad (3.23)
\]

\[
U_i^k(b_i h) = U^i(b_1(U_{k-1}^i(b_i h))_{-a}).
\]

In accordance with (2.19) we can write for \( k = n - p + 1 \)

\[
(U^i(b_i h))(t) = \int_0^t \int_0^{t_{n-A(t_n)}} \cdots \int_0^{t_{p+1-A(t_{p+1})}} [k_i(t, t_n)b_i(t_n)k_i(t_n - A(t_n), t_n-1)] \cdots \\
\cdots b_1(t_{p+1})k_i(t_{p+1} - A(t_{p+1}), t_p)b_1(t_p)h(t_p)] \, dt_p \, dt_{p+1} \cdots \, dt_n. \quad (3.24)
\]

Let us introduce the constants \( g, f_q, b_q \) defined by

\[
g = \max\{|g(t)|, \ t \in E_0\}, \quad f_q = \max\{|f(t)|, \ 0 \leq t \leq \gamma_q(0)\}, \quad b_q = \max\{|b_1(t)|, \ 0 \leq t \leq \gamma_q(0)\}.
\]

Because of \( \inf_{t \in [0,a]} A(t) = d > 0 \) we have

\[
\int_0^{t-A(t)} \int_0^{t_n-A(t_n)} \cdots \int_0^{t_{p+1-A(t_{p+1})}} \leq \int_0^{t-d} \int_0^{t_n-d} \cdots \int_0^{t_{p+1-d}}
\]

and we shall use the following equality:

\[
\int_0^{t-d} \int_0^{t_n-d} \cdots \int_0^{t_{p+1-d}} \, dt_p \, dt_{p+1} \cdots \, dt_n = \frac{(t - d)(t - (n - p + 2)d)^{n-p}}{(n - p + 1)!}.
\]

In particular, for \( t = (n + 2)d \) we have

\[
\int_0^{(n+1)d} \int_0^{t_n-d} \cdots \int_0^{t_{p+1-d}} \, dt_p \, dt_{p+1} \cdots \, dt_n = \frac{(n + 1)pq^{n-p+1}}{n - p + 1)!} = I_{np}. \quad (3.25)
\]
From (2.19) and (3.24) if \( k = n - p + 1 \), we can write

\[
(U_k(b_1h)(t)) - (U_k^j(b_1z)(t)) = \\
\int_0^{t - \Delta(t)} \int_0^{t - \Delta(t_n)} \cdots \int_0^{t - \Delta(t_{n-1})} \{[k(t, t_n) - k_i(t, t_n)]b_1(t_n)k(t_n - \Delta(t_n), t_{n-1}) \cdots \\
\cdots b_1(t_{p+1})k_i(t_{p+1} - \Delta(t_{p+1}), t_p)b_1(t_p)z(t_p) \\
+ \sum k(t, t_n)b_1(t_n)k(t_n - \Delta(t_n), t_{n-1}) \cdots \\
\cdots b_1(t_j)[k(t_j - \Delta(t_j), t_{j-1}) - k_i(t_j - \Delta(t_j), t_{j-1})]b_1(t_{j-1}) \cdots \\
\cdots b_1(t_{p+1})k_i(t_{p+1} - \Delta(t_{p+1}), t_p)b_1(t_p)z(t_p) \\
+ k(t, t_n)b_1(t_n)k(t_n - \Delta(t_n), t_{n-1}) \cdots \\
\cdots b_1(t_{p+1})k(t_{p+1} - \Delta(t_{p+1}), t_p)b_1(t_p)[h(t_p) - z(t_p)] \rangle dt_p \, dt_{p+1} \cdots \, dt_n. \tag{3.26}
\]

Now, if we use these to make operation easy;

\[
b_h \equiv b_q, \quad S_h \equiv S_q, \quad \gamma_{ij} \equiv \gamma_{iq}, \quad M_{ih} \equiv M_{iq}.
\]

Here, there are \( j, h = n, n-1, \ldots, p+1, p \) for \( p = n - k + 1, k \geq 1 \). In addition,

\[
|k(t, t_m) - k_i(t, t_m)| \leq 2S_n^2 \gamma_{im} \\
|k(t, t_m)| \leq 2S_n^2 \\
|k_i(t, t_m)| \leq 2S_n^2(1 + E_i)M_{im} \\
|b_1(t_m)| \leq b_m \\
\forall m \in \{n, n-1, \ldots, p+1, p\}.
\]

If we denote

\[
\alpha_1 = \max\{|z(t_p)|, \gamma_{p-1}(0) \leq t_p \leq \gamma_{p}(0)\}, \\
\alpha_2 = \max\{|h(t_p) - z(t_p)|, \gamma_{p-1}(0) \leq t_p \leq \gamma_{p}(0)\},
\]

then from (3.25) and (3.26) for \( t \in [\gamma_{n-1}(0), \gamma_{n}(0)] \) it follows that:

\[
|U_k(b_1h)(t) - U_k^j(b_1z)(t)| \leq 2^k I_{sp} \prod_{h=p}^{n} b_h S_h^2 \left\{ \alpha_1 \sum_{j=p}^{n} \gamma_{ij} \prod_{h=p}^{j-1} M_{ih}(1 + E_i)^{(j-p)} + \alpha_2 \right\}. \tag{3.27}
\]

If \( p = 0 \) and \( h = z = g_{-A} \), the expression (3.27) takes on the form \( \alpha_2 = 0 \) and

\[
|U_{n+1}(b_1g_{-A}) - U_{n+1}^j(b_1g_{-A})| \leq \Sigma_{n,i},
\]

\[
\Sigma_{n,i} = 2^{n+1} \alpha_{n0} s \prod_{h=0}^{n} b_h S_h^2 \sum_{j=0}^{n} \gamma_{ij} \prod_{h=0}^{j-1} M_{ih}(1 + E_i)^{j} \rightarrow 0, \quad i \rightarrow \infty.
\]

If we consider \( z = \Phi_{-A}, h = \Phi_{-A}^j \), from (2.14), (3.3), and (3.13) for \( t_p \in [\gamma_{p-1}(0), \gamma_{p}(0)] \) it follows that:
and from (3.27) we can write
\[ |U_k(b_1 \Phi_{-A}) - U_k'(b_1 \Phi'_{-A})| \leq Y_{k,i}, \]
\[ Y_{k,i} = 2^k I_{n_p}(|g(0)| + |g'(0)|) \prod_{h=p}^{n} b_h S_h^2 S_{p-1} \left\{ \sum_{j=p}^{n} \gamma_{ij} \prod_{h=p}^{j-1} M_{ih}(1 + E_i)^{(j-p)} + E_i \right\} \]
\[ \to 0, \quad i \to \infty. \]

Taking \( h = (U^f)_{-A}, z = (Uf)_{-A} \) and using that \( t_p \in [\gamma_{p-1}(0), \gamma_p(0)] \), from (2.9), (3.21), and (3.23) it follows that:
\[ |(Uf)(t_p - \Delta(t_p)) - (U^f(t_p - \Delta(t_p)))| \leq 2S_{p-1}^2 \gamma_{i,p-1} f_{p-1}(t_p - \Delta(t_p)) \]
\[ \leq 2S_{p-1}^2 \gamma_{i,p-1} f_{p-1} \gamma_p(0) \quad (3.28) \]
and from (3.23) and (3.22),
\[ |(Uf)(t_p - \Delta(t_p))| \leq 2\gamma_p(0)S_{p-1}^2 f_{p-1}. \quad (3.29) \]

From (3.27), (3.28), and (3.29) we can write
\[ |U_k(b_1(Uf)_{-A})(t_k) - U_k'(b_1(U^f)_{-A})(t_k)| \leq U_{k,i}, \]
where
\[ U_{k,i} = 2^{k+1} I_{n_p} S_{p-1} f_{p-1} \gamma_p(0) \prod_{h=p}^{n} b_h S_h^2 \left\{ \sum_{j=p}^{n} \gamma_{ij} \prod_{h=p}^{j-1} M_{ih}(1 + E_i)^{(j-p)} + \gamma_{i,p-1} \right\} \]
\[ \to 0, \quad i \to \infty. \]

If we denote by \( x'(t) \) the approximate solution of (1.1) defined in \([\gamma_{n-1}(0), \gamma_n(0)]\) by
\[ x' = \Phi' + U^f + \sum_{k=1}^{n} (-1)^k U_k'(b_1(U^f)_{-A}) + \sum_{k=1}^{n} (-1)^k U_k(b_1 \Phi'_{-A}) \]
\[ + (-1)^{n+1} U_{n+1}(b_1 g_{-A}) \quad (3.30) \]
from (2.18) and (3.30) and the above estimates it follows that the error \( x(t) - x'(t) \) for \( i \geq i_0 \) and \( t \in [\gamma_{n-1}(0), \gamma_n(0)] \subset [0, a) \) satisfies
\[ |x(t) - x'(t)| \leq |\Phi(t) - \Phi'(t)| + |Uf(t) - Uf'(t)| + \sum_{k=1}^{n} |U_{k}(b_{1}(Uf)_{-A})(t)| \\
- U'_{k}(b_{1}(Uf)_{-A})(t) + \sum_{k=1}^{n} |U_{k}(b_{1}\Phi_{-A})(t) - U'_{k}(b_{1}\Phi_{-A})(t)| \\
+ |U_{n+1}(b_{1}g) - U'_{n+1}(b_{1}g)| \\
\leq (|g'(0)| + |g'(0)|)S_{n}E_{i} + 2\gamma_{n}(0)S_{n}^{2}\gamma_{in}f_{n} + \sum_{k=1}^{n} (U_{k,i} + Y_{k,i}) + \mathfrak{I}_{n,i}. \tag{3.31} \]

Thus, for a fixed interval \([\gamma_{n-1}(0), \gamma_{n}(0)]\) and an admissible error \(\varepsilon\), to construct a finite analytic approximate solution whose error be smaller than \(\varepsilon\) in \([\gamma_{n-1}(0), \gamma_{n}(0)]\) it suffices to take \(i \geq i_{0}\) so large that

\[ (|g'(0)| + |g'(0)|)S_{n}E_{i} + 2\gamma_{n}(0)S_{n}^{2}\gamma_{in}f_{n} + \sum_{k=1}^{n} (U_{k,i} + Y_{k,i}) + \mathfrak{I}_{n,i} < \varepsilon. \tag{3.32} \]

This is possible since \(E_{i}, \gamma_{in}, \mathfrak{I}_{n,i}, U_{k,i}\) and \(Y_{k,i}\) tend to 0 as \(i \to \infty\). Hence the following result has been proved.

**Theorem 3.** Consider problem (1.1) under the hypotheses of Theorem 2 using the previous notation. If \(x'(t)\) is the function defines by (3.30), then \(x'(t)\) converges to the exact solution \(x(t)\) of (1.1) as \(i \to \infty\) for any \(t \in [0, a]\). If \(n\) is a positive integer such that \(\gamma_{n}(0) < a\), then the error of the approximate solution \(x'(t)\) with respect to the exact solution \(x(t)\) satisfies (3.31) for \(t \in [\gamma_{n-1}(0), \gamma_{n}(0)]\). Given an admissible error \(\varepsilon > 0\), taking \(i \geq i_{0}\) satisfying (3.32), one gets an approximation whose error is bounded above by \(\varepsilon\) for \(t \in [\gamma_{n-1}(0), \gamma_{n}(0)]\).

**References**


