Stability properties of second order delay integro-differential equations

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A basic theorem on the behavior of solutions of scalar linear second order delay integro-differential equations is established. As a consequence of this theorem, a stability criterion is obtained.

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1. Introduction and preliminaries

The theory of Volterra integral and integro-differential equations is interesting in itself and the application of this theory is rapidly increasing to various fields. For the basic theory of integral equations, we choose to refer to the books by Burton [1], Corduneanu [2] and Miller [3]. The theory of delay differential equations is of both theoretical and practical interest. For the basic theory of delay differential equations, the reader is referred to the books by Bellman and Cooke [4], El'sgol'ts and Norkin [5], Hale and Verduyn Lunel [6], Kolmanovskii [7] and Lakshmikantham, Wen and Zhang [8].

In this paper, we will give a basic theorem on the behavior of solutions of scalar linear second order delay integro-differential equations. An asymptotic result for the solutions is obtained. Also, an estimate of the solutions is established. The sufficient conditions for the stability, the asymptotic stability and instability of the trivial solution and some examples are given. Our results are derived by the use of real roots (with an appropriate property) of the corresponding (in a sense) characteristic equations.

The very interesting asymptotic and stability results were given by Kordonis and Philos [9] and Philos and Purnaras [10]. The techniques applied in [11] are originated in a combination of the methods used in [9] and [10].

Yeniçerioğlu [11] obtained some results on the qualitative behavior of the solutions of a second order linear autonomous delay differential equation with a single delay. The main idea in [11] is that of transforming the second order delay differential equation into a first order delay differential equation, by the use of a real root of the corresponding characteristic equation. The same idea will be used in this paper to obtain some general results.

Koto [12] and Zhao, Xu and Liu [13] studied with the stability of numerical methods for delay integro-differential equations and linear neutral Volterra delay integro-differential system. Equation of form of (1) can be used as test of equations for numerical methods (see [12,13]).

In [14], it has been established the application of the so-called mixed interpolation methods for linear Volterra integro-differential equations of the form

\[ y^{(r)}(x) = f(x) + ay(x) + b \int_{-\infty}^{x} K(x - s)y(s)ds \]
with \( r = 1 \) or \( r = 2 \). Here \( a \) and \( b \) are constant, \( K \in L^1(0, \infty) \) and \( f \) is a continuous, periodic function. Equation of form of (1) with this method can solve.


Finally, Hale and Verduyn Lunel [6, p.149] have established the stability criteria for a second order delay integro-differential equation

\[
A\ddot{x}(t) + B\dot{x}(t) = \int_0^t F(\theta)x(t - \theta)d\theta,
\]

where \( A, B \) and \( F \) are symmetric \( n \times n \) matrices and \( F \) is continuously differentiable. This equation is obtained the stability of second order delay integro-differential equation using Lyapunov function. However, we study the stability of the some problem using the method of characteristic roots.

Let us consider initial value problem for second order delay integro-differential equation

\[
x''(t) + ax(t) = \int_0^t f(s)x(t - s)ds, \quad t \geq 0,
\]

\[
x(t) = \phi(t), \quad -r \leq t \leq 0,
\]

where \( a \) is a real number, \( r \) is positive real number, \( f \) is a continuous real-valued function on the interval \([0, \infty)\) and \( \phi(t) \) is a given continuously differentiable initial function on the interval \([-r, 0]\).

As usual, a twice continuously differentiable real-valued function \( x \) defined on the interval \([-r, \infty)\) is said to be a solution of the initial value problem (1) and (2) if \( x \) satisfies (1) for all \( t \geq 0 \) and (2) for all \(-r \leq t \leq 0\).

It is known that, for any given initial function \( \phi \), there exists a unique solution of the initial problem (1) and (2) or, more briefly, the solution of (1) and (2).

If we look for a solution of (1) of the form \( x(t) = e^{\lambda t} \) for \( t \in IR \), we see that \( \lambda \) is a root of the characteristic equation

\[
\lambda^2 + a = \int_0^r f(s)e^{-\lambda s}ds.
\]

For a given real root \( \lambda_0 \) of the characteristic equation (3), we consider the first order delay integro-differential equation

\[
z''(t) = -2\lambda_0 z(t) - \int_0^t f(s)e^{-\lambda_0 s}\left\{ \int_{s}^{t} z(\tau)d\tau \right\} ds.
\]

A solution of the delay integro-differential equation (4) is a continuous real-valued function \( z \) defined on the interval \([-r, \infty)\), which is continuously differentiable on \([0, \infty)\) and satisfies (4) for all \( t \geq 0 \).

The second characteristic equation of the delay integro-differential equation (4) is

\[
\delta = -2\lambda_0 - \delta^{-1} \int_0^r f(s)e^{-\lambda_0 s}\left( 1 - e^{-\delta s} \right) ds.
\]

This equation is obtained from (4) by seeking solutions of the form \( z(t) = e^{\delta t} \) for \( t \in IR \).

For our convenience, we introduce some notations. For a given real root \( \lambda_0 \) of the characteristic equation (3), we set

\[
\beta_{\lambda_0} = 2\lambda_0 + \int_0^r f(s)e^{-\lambda_0 s}ds
\]

and, also, we define

\[
L(\lambda_0; \phi) = \phi'(0) + \lambda_0 \phi(0) + \int_0^r f(s)e^{-\lambda_0 s}\left\{ \int_{s}^{0} e^{-\lambda_0 \tau} \phi(\tau)d\tau \right\} ds.
\]

We will now give a proposition, which plays a crucial role in obtaining our main results.

**Proposition.** Let \( \lambda_0 \) be a real root of the characteristic equation (3), and let \( \beta_{\lambda_0} \) and \( L(\lambda_0; \phi) \) be defined by (6) and (7), respectively. Suppose that \( \beta_{\lambda_0} \neq 0 \).

Then a continuous real-valued function \( x \) defined on the interval \([-r, \infty)\) is the solution of (1) and (2) if and only if the function \( z \) defined by

\[
z(t) = e^{-\lambda_0 t}x(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}}, \quad \text{for} \ t \geq -r
\]

is the solution of the delay integro-differential equation (4) which satisfies the initial condition

\[
z(t) = e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}}, \quad \text{for} \ -r \leq t \leq 0.
\]
**Proof.** Let \( x \) be the solution of (1) and (2), and define
\[
y(t) = e^{-\lambda_0 t} x(t), \quad \text{for } t \in [-r, \infty).
\]

Then, by taking into account the fact that \( \lambda_0 \) is a real root of the characteristic equation (3), for every \( t \geq 0 \), we have
\[
x''(t) + ax(t) - \int_0^t f(s)x(t-s)ds = e^{\lambda_0 t} \left\{ y''(t) + 2\lambda_0 y'(t) + \lambda_0^2 y(t) + ay(t) - \int_0^t f(s)e^{-\lambda_0 s}y(t-s)ds \right\}
\[
= e^{\lambda_0 t} \left\{ y'(t) + 2\lambda_0 y(t) \right\}' + (\lambda_0^2 + a)y(t) - \int_0^t f(s)e^{-\lambda_0 s}y(t-s)ds.
\]

Hence, the fact that \( x \) is a solution of second order delay integro differential equation (1) is equivalent to the fact that \( y \) satisfies
\[
\left[ y'(t) + 2\lambda_0 y(t) \right]' = - \left( \lambda_0^2 + a \right) y(t) + \int_0^t f(s)e^{-\lambda_0 s}y(t-s)ds.
\]

for all \( t \geq 0 \). On the other hand, \( x \) satisfies the initial condition (2) if and only if \( y \) satisfies the initial condition
\[
y(t) = e^{-\lambda_0 t} \phi(t), \quad \text{for } t \in [-r, 0].
\]

Furthermore, by using the fact that \( \lambda_0 \) is a real root of (3) and taking into account (11), we can verify that (10) is equivalent to
\[
y'(t) + 2\lambda_0 y(t) = y'(0) + 2\lambda_0 y(0) - \left( \lambda_0^2 + a \right) \int_0^t y(s)ds + \int_0^t f(s)e^{-\lambda_0 s} \left\{ \int_0^s y(u-s)du \right\} ds,
\]
\[
y'(t) = -2\lambda_0 y(t) + \phi'(0) + \lambda_0 \phi(0) - \left( \lambda_0^2 + a \right) \int_0^t y(s)ds + \int_0^t f(s)e^{-\lambda_0 s} \left\{ \int_{t-s}^t y(\tau)d\tau \right\} ds,
\]
\[
y'(t) = -2\lambda_0 y(t) - \left( \lambda_0^2 + a \right) \int_0^t y(s)ds + \int_0^t f(s)e^{-\lambda_0 s} \left\{ \int_0^s y(\tau)d\tau \right\} ds + L(\lambda_0; \phi),
\]
\[
y'(t) = -2\lambda_0 y(t) - \int_0^t f(s)e^{-\lambda_0 s} \left\{ \int_0^s y(\tau)d\tau \right\} ds + \int_0^t f(s)e^{-\lambda_0 s} \left\{ \int_0^s y(\tau)d\tau \right\} ds + L(\lambda_0; \phi),
\]
\[
y'(t) = -2\lambda_0 y(t) - \int_0^t f(s)e^{-\lambda_0 s} \left\{ \int_0^s y(\tau)d\tau \right\} ds + L(\lambda_0; \phi),
\]

for all \( t \geq 0 \).

Now, we take into account the assumption \( \beta_{\lambda_0} \neq 0 \) and we define
\[
z(t) = y(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}}, \quad \text{for } t \geq -r.
\]

Then, because of the definition of \( \beta_{\lambda_0} \) by (6), it is a matter of elementary calculations to show that \( y \) satisfies (12) for \( t \geq 0 \) if and only if \( z \) satisfies (4) for all \( t \geq 0 \), i.e., if and only if \( z \) is a solution of the delay integro-differential equation (4). Moreover, we see that the initial condition (11) is equivalent written as follows
\[
z(t) = e^{-\lambda_0 t} \phi(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}}, \quad \text{for } -r \leq t \leq 0.
\]

The proof of our proposition is complete. \( \square \)

Before closing this section, we will give three well-known definitions. The trivial solution \( x \equiv 0 \) is a trivial solution of initial value problem (1) of (1) is said to be “stable” if for every \( \epsilon > 0 \), there exists a number \( \ell = \ell(\epsilon) > 0 \) such that, for any continuously differentiable initial function \( \phi \) with
\[
\| \phi \| = \max_{-r \leq t \leq 0} |\phi(t)| < \ell,
\]
the solution \( x \) of (1) and (2) satisfies
\[
|x(t)| < \epsilon, \quad \text{for all } t \in [-r, \infty).
\]

Otherwise, the trivial solution of (1) is said to be “unstable”. Moreover, the trivial solution of (1) is called “asymptotically stable” if it is stable in the above sense and in addition there exists a number \( \ell_0 > 0 \) such that, for any continuously differentiable initial function \( \phi \) with \( \| \phi \| < \ell_0 \), the solution \( x \) of (1) and (2) satisfies
\[
\lim_{t \to \infty} x(t) = 0.
\]

Let \( C^1([-r, 0], IR) \) be the set of all continuously differentiable real-valued functions on the interval \([-r, 0]\).
2. Statement of the main results and comments

Our purpose in this section is to establish the following basic theorem

**Theorem 1.** Let \( \lambda_0 \) and \( \delta_0 \) be real roots of the characteristic equations (3) and (5), respectively. Suppose that \( \beta_{\lambda_0} \neq 0 \). (Note that, because \( \beta_{\lambda_0} \neq 0 \), we always \( \delta_0 \neq 0 \).) Set

\[
\eta_{\lambda_0, \delta_0} \equiv 1 - \delta_0^{-2} \int_{0}^{r} f(s)e^{-\lambda_0 s} (1 - e^{-\delta_0 s} - \delta_0 s e^{-\delta_0 s}) \, ds.
\]

and, also, define

\[
R(\lambda_0, \delta_0; \phi) = \phi(0) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} - \int_{0}^{r} f(s)e^{-\lambda_0 s} \left\{ \int_{0}^{s} e^{-\delta_0 \tau} \left( \int_{-\tau}^{0} e^{-\lambda_0 u} \phi(u) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} \right) \, du \right\} \, ds.
\]

Assume that the roots \( \lambda_0 \) and \( \delta_0 \) have the following property

\[
\mu_{\lambda_0, \delta_0} \equiv \delta_0^{-2} \int_{0}^{r} \left| f(s)e^{-\lambda_0 s} \right| \left\{ \int_{0}^{s} e^{-\delta_0 \tau} \, d\tau \right\} \, ds < 1.
\]

(This assumption guarantees that \( \eta_{\lambda_0, \delta_0} > 0 \).

Then, for any \( \phi \in C^1([-r, 0], IR) \), the solution \( x \) of (1) and (2) satisfies

\[
\left| e^{-(\lambda_0 + \delta_0) t} x(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} e^{-\lambda_0 t} - \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}} \right| \leq M(\lambda_0, \delta_0; \phi) \mu_{\lambda_0, \delta_0}, \quad \text{for all } t \geq 0,
\]

where \( L(\lambda_0; \phi) \) was given in (7) and

\[
M(\lambda_0, \delta_0; \phi) = \max_{-r \leq t \leq 0} \left| e^{-(\lambda_0 + \delta_0) t} \phi(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} e^{-\lambda_0 t} - \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}} \right|.
\]

Before we prove the above theorem, we will present some observations, which are concerned with a real root \( \lambda_0 \) of the characteristic equation (3) and a real root \( \delta_0 \) of the characteristic equation (5).

Let \( F(\delta) \) denote the characteristic function of (5), i.e.,

\[
F(\delta) = \delta + 2\lambda_0 + \delta^{-1} \int_{0}^{r} f(s)e^{-\lambda_0 s} (1 - e^{-\delta_0 s}) \, ds.
\]

Since \( \delta = 0 \) is a removable singularity of \( F(\delta) \), we can regard \( F(\delta) \) as a entire function with

\[
F(0) = 2\lambda_0 + \int_{0}^{r} f(s)e^{-\lambda_0 s} \, ds = \beta_{\lambda_0}.
\]

But, by the definition of \( \beta_{\lambda_0} \neq 0 \), a root of the characteristic equation (5) must become \( \delta_0 \neq 0 \). Define \( \mu_{\lambda_0, \delta_0} \) by (15). It is clear \( \mu_{\lambda_0, \delta_0} \) is positive. So, (15) can equivalently be written as follows

\[
0 < \mu_{\lambda_0, \delta_0} < 1.
\]

Furthermore, for the real constant \( \eta_{\lambda_0, \delta_0} \) defined by (13), we have

\[
|\eta_{\lambda_0, \delta_0} - 1| = \left| \delta_0^{-2} \int_{0}^{r} f(s)e^{-\lambda_0 s} (1 - e^{-\delta_0 s} - \delta_0 s e^{-\delta_0 s}) \, ds \right|
\]

\[
\leq \delta_0^{-2} \int_{0}^{r} |f(s)e^{-\lambda_0 s} (1 - e^{-\delta_0 s} - \delta_0 s e^{-\delta_0 s})| \, ds \equiv \mu_{\lambda_0, \delta_0}.
\]

That is,

\[
|\eta_{\lambda_0, \delta_0} - 1| \leq \mu_{\lambda_0, \delta_0}.
\]

Thus, if we assume that (15) is satisfied, i.e., that (18) holds, then (19) gives \( |\eta_{\lambda_0, \delta_0} - 1| < 1 \). This guarantees, in particular, that \( \eta_{\lambda_0, \delta_0} > 0 \).

**Proof of Theorem 1.** Let \( x \) be the solution of (1) and (2). Define the function \( z \) by (8). By Proposition, the fact that \( x \) is the solution of the of (1) and (2) is equivalent to the fact that \( z \) is the solution of the delay integro-differential equation (4) which...
satisfies the initial condition (9), \( \delta_0 \) be a real root of the characteristic equation (5). Define for \( \delta_0 \neq 0 \)

\[
v(t) = e^{-\delta_0 t}z(t), \quad \text{for all } t \in [-r, \infty).
\]

Then, from (4), we obtain, for every \( t \geq 0 \),

\[
v'(t) = -(2\lambda_0 + \delta_0)v(t) - \int_0^t f(s)e^{-\lambda_0 s} \left\{ \int_0^s e^{-\delta_0 \tau} v(t - \tau) \, d\tau \right\} \, ds.
\]

Moreover, the initial condition (9) can be equivalently written

\[
v(t) = \phi(t)e^{-(\lambda_0 + \delta_0)t} - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}}, \quad \text{for } t \in [-r, 0].
\]

Furthermore, by using the fact that \( \delta_0 \neq 0 \) is a real root of (5) and taking into account (21), we can verify that (20) is equivalent to

\[
v(t) = v(0) - (2\lambda_0 + \delta_0) \int_0^t v(s) \, ds - \int_0^t f(s)e^{-\lambda_0 s} \left\{ \int_0^s e^{-\delta_0 \tau} \left( \int_0^t v(u - \tau) \, du \right) \, d\tau \right\} \, ds,
\]

\[
v(t) = \phi(0) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} - (2\lambda_0 + \delta_0) \int_0^t v(s) \, ds - \int_0^t f(s)e^{-\lambda_0 s} \left\{ \int_0^s e^{-\delta_0 \tau} \left( \int_0^t v(u) \, du \right) \, d\tau \right\} \, ds,
\]

\[
v(t) = R(\lambda_0, \delta_0; \phi) - (2\lambda_0 + \delta_0) \int_0^t v(s) \, ds - \int_0^t f(s)e^{-\lambda_0 s} \left\{ \int_0^s e^{-\delta_0 \tau} \left( \int_0^t v(u) \, du \right) \, d\tau \right\} \, ds,
\]

\[
v(t) = R(\lambda_0, \delta_0; \phi) + \int_0^t f(s)e^{-\lambda_0 s} \left\{ \int_0^s e^{-\delta_0 \tau} \left( \int_0^t v(u) \, du \right) \, d\tau \right\} \, ds,
\]

\[
v(t) = R(\lambda_0, \delta_0; \phi) + \int_0^t f(s)e^{-\lambda_0 s} \left\{ \int_0^s e^{-\delta_0 \tau} \left( \int_0^t v(u) \, du \right) \, d\tau \right\} \, ds.
\]

Next, we define

\[
w(t) = v(t) - \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}}, \quad \text{for } t \geq -r.
\]

Then we can see that (22) reduces to the following equivalent equation

\[
w(t) = \int_0^t f(s)e^{-\lambda_0 s} \left\{ \int_0^s e^{-\delta_0 \tau} \left( \int_{t-\tau}^t v(u) \, du \right) \, d\tau \right\} \, ds.
\]

On the other hand, the initial condition (21) can be equivalently written

\[
w(t) = \phi(t)e^{-(\lambda_0 + \delta_0)t} - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} e^{-\delta_0 t} - \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}} \quad \text{for } t \in [-r, 0].
\]

Applying the definitions of \( y, z, v \) and \( w \) we can obtain that (16) is equivalent to

\[
|w(t)| \leq M(\lambda_0, \delta_0; \phi) \mu_{\lambda_0, \delta_0}, \quad \forall t \geq 0.
\]

So, we will prove (26).

From (17) and (25) it follows that

\[
|w(t)| \leq M(\lambda_0, \delta_0; \phi), \quad \text{for } t \in [-r, 0].
\]

We will show that \( M(\lambda_0, \delta_0; \phi) \) is a bound of \( w \) on the whole interval \([-r, \infty)\). Namely

\[
|w(t)| \leq M(\lambda_0, \delta_0; \phi), \quad \text{for all } t \in [-r, \infty).
\]
To this end, let us consider an arbitrary number \( \varepsilon > 0 \). We claim that
\[
|w(t)| < M(\lambda_0, \delta_0; \phi) + \varepsilon, \quad \text{for every } t \in [-r, \infty).
\] (29)

Otherwise, by (27), there exists a \( t^* > 0 \) such that
\[
|w(t)| < M(\lambda_0, \delta_0; \phi) + \varepsilon, \quad \text{for } t < t^* \quad \text{and} \quad |w(t^*)| = M(\lambda_0, \delta_0; \phi) + \varepsilon.
\]

Then using (24), we obtain
\[
M(\lambda_0, \delta_0; \phi) + \varepsilon = |w(t^*)| \leq \int_{0}^{t^*} |f(s)|e^{-\lambda_0 s} \left\{ \int_{0}^{s} e^{-\delta_0 \tau} \left( \int_{t^*}^{t} |w(u)|\,du \right) \,d\tau \right\} \,ds 
\leq \delta_0^{-2} \int_{0}^{t^*} |f(s)|e^{-\lambda_0 s} \left( 1 - e^{-\delta_0 s} - \delta_0 se^{-\delta_0 s} \right) \,ds (M(\lambda_0, \delta_0; \phi) + \varepsilon) < M(\lambda_0, \delta_0; \phi) + \varepsilon,
\]

which, in view of (15), leads to a contradiction. So, our claim is true. Since (29) holds for every \( \varepsilon > 0 \), it follows that (28) is always satisfied. By using (28) and (24), we derive
\[
|w(t)| \leq \int_{0}^{t} |f(s)|e^{-\lambda_0 s} \left\{ \int_{0}^{s} e^{-\delta_0 \tau} \left( \int_{t}^{t} |w(u)|\,du \right) \,d\tau \right\} \,ds
\leq M(\lambda_0, \delta_0; \phi) \int_{0}^{t} |f(s)|e^{-\lambda_0 s} \left\{ \int_{0}^{s} e^{-\delta_0 \tau} \,d\tau \right\} \,ds = M(\lambda_0, \delta_0; \phi)\mu_{\lambda_0, \delta_0},
\]
for all \( t \geq 0 \). That means (26) holds.

The proof of the Theorem 1 is completed. \( \square \)

**Theorem 2.** Let \( \lambda_0 \) and \( \delta_0 \) be real roots of the characteristic equations (3) and (5), respectively. Suppose that \( \beta_{\lambda_0} \neq 0 \), where \( \beta_{\lambda_0} \) is defined by (6). Then, for any \( \phi \in C^{1}([-r, 0], IR) \), the solution \( x \) of (1) and (2) satisfies
\[
\lim_{t \to \infty} \left\{ e^{-(\lambda_0 + \beta_0)t}x(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} e^{-\delta_0 t} \right\} = \frac{R(\lambda_0; \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}},
\]
where \( L(\lambda_0; \phi) \), \( \eta_{\lambda_0, \delta_0} \) and \( R(\lambda_0; \delta_0; \phi) \) were given in (7), (13) and (14), respectively.

**Proof.** By the definitions of \( y, z, v \) and \( w \), we have to prove that
\[
\lim_{t \to \infty} w(t) = 0.
\] (30)

In the end of the proof we will establish (30). By using (24) and taking into account (26) and (28), one can show, by an easy induction, that \( w \) satisfies
\[
|w(t)| \leq \left( \mu_{\lambda_0, \delta_0} \right)^n M(\lambda_0, \delta_0; \phi), \quad \text{for all } t \geq nr - r, \, (n = 0, 1, \ldots).
\] (31)

But, (15) guarantees that \( 0 < \mu_{\lambda_0, \delta_0} < 1 \). Thus, from (31) it follows immediately that \( w \) tends to zero as \( t \to \infty \), i.e. (30) holds.

The proof of the Theorem 2 is completed. \( \square \)

**Theorem 3.** Let \( \lambda_0 \) and \( \delta_0 \) be real roots of the characteristic equations (3) and (5), respectively. Suppose that \( \beta_{\lambda_0} \neq 0 \), where \( \beta_{\lambda_0} \) is defined by (6). (Note that, because of \( \beta_{\lambda_0} \neq 0 \), we always \( \delta_0 \neq 0 \) and assume that \( \mu_{\lambda_0, \delta_0} < 1 \), where \( \mu_{\lambda_0, \delta_0} \) is defined by (15). Then, for any \( \phi \in C^{1}([-r, 0], IR) \), the solution \( x \) of (1) and (2) satisfies for all \( t \geq 0 \)
\[
x(t) \leq \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} N(\lambda_0, \delta_0; \phi)e^{\delta_0 t} + \left[ \frac{h_{\lambda_0, \delta_0}}{\eta_{\lambda_0, \delta_0}} + \left( 1 + \frac{k_0e_0}{|\beta_{\lambda_0}|} + \frac{h_{\lambda_0, \delta_0}}{\eta_{\lambda_0, \delta_0}} \right) \mu_{\lambda_0, \delta_0} \right] N(\lambda_0, \delta_0; \phi)e^{(\lambda_0 + \beta_0)t},
\] (32)

where
\[
\eta_{\lambda_0, \delta_0} \text{ was given in (13)},
\]
\[
k_{\lambda_0} = 1 + |\lambda_0| + \int_{0}^{t} |f(s)|e^{-\lambda_0 s}ds,
\] (33)
\[
h_{\lambda_0, \delta_0} = 1 + \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} + \frac{\delta_0}{2} \int_{0}^{t} |f(s)|e^{-\lambda_0 s} \left( 1 - e^{-\delta_0 s} \right) \left( 1 - \frac{1}{|\beta_{\lambda_0}|} \right) + \delta_0 s \left( \frac{1}{|\beta_{\lambda_0}|} - e^{-\delta_0 s} \right) \,ds,
\] (34)
\[
e_0 = \max_{-r \leq s \leq 0} \{ e^{-\delta_0 s} \}.
\] (35)
and

\[ N(\lambda_0, \delta_0; \phi) = \max \left\{ \max_{-T \leq t \leq 0} |e^{-\lambda_0 t} \phi(t)|, \max_{-T \leq t \leq 0} |e^{-(\lambda_0 + \delta_0) t} \phi(t)|, \max_{-T \leq t \leq 0} |\phi(t)|, \max_{-T \leq t \leq 0} |\phi(t)| \right\}. \] (36)

**Corollary.** Let \( \lambda_0 \) be a real root of the characteristic equation (3), and suppose that \( \beta_{\lambda_0} \neq 0 \), where \( \beta_{\lambda_0} \) is defined by (6). Furthermore, let \( \delta_0 \) be a real root of the characteristic equation (5). (Note that, because of \( \beta_{\lambda_0} \neq 0 \), we always have \( \delta_0 \neq 0 \).)

Assume that (15) holds. Then the trivial solution of (1) is stable if \( \lambda_0 \leq 0, \lambda_0 + \delta_0 \leq 0 \), it is asymptotically stable if \( \lambda_0 < 0, \lambda_0 + \delta_0 < 0 \) and it is unstable if \( \delta_0 > 0, \lambda_0 + \delta_0 > 0 \).

**Proof of Theorem 3.** By Theorem 1, (16) is satisfied, where \( L(\lambda_0; \phi), R(\lambda_0, \delta_0; \phi) \) and \( M(\lambda_0, \delta_0; \phi) \) are defined by (7), (14) and (17), respectively. From (16) it follows that

\[ e^{-(\lambda_0 + \delta_0)t} |\phi(t)| \leq \frac{|L(\lambda_0; \phi)|}{|\beta_{\lambda_0}|} e^{-\lambda_0 t} + \frac{|R(\lambda_0, \delta_0; \phi)|}{\eta_{\lambda_0, \delta_0}} + M(\lambda_0, \delta_0; \phi) \mu_{\lambda_0, \delta_0}. \] (37)

Furthermore, by using (33)–(36), from (7), (14) and (17), we obtain

\[
\begin{align*}
|L(\lambda_0; \phi)| &\leq |\phi(0)| + |\lambda_0| |\phi(0)| + \int_0^t |f(s)| e^{-\lambda_0 s} \left\{ \int_{-s}^0 e^{-\lambda_0 \tau} \left| \phi(\tau) \right| d\tau \right\} ds \\
&\leq \left( 1 + |\lambda_0| + \int_0^t |f(s)| e^{-\lambda_0 s} ds \right) N(\lambda_0, \delta_0; \phi) = k_{\lambda_0} N(\lambda_0, \delta_0; \phi), \\
|R(\lambda_0, \delta_0; \phi)| &\leq |\phi(0)| + \frac{|L(\lambda_0; \phi)|}{|\beta_{\lambda_0}|} \\
&+ \int_0^t |f(s)| e^{-\lambda_0 s} \left\{ \int_0^s e^{-\lambda_0 \tau} \left( \int_0^\tau e^{-\lambda_0 u} \left| \phi(u) \right| du \right) + \frac{|L(\lambda_0; \phi)|}{|\beta_{\lambda_0}|} \right\} ds \leq \left( 1 + \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} + \delta_0^2 \int_0^t |f(s)| e^{-\lambda_0 s} \left( 1 - e^{-\lambda_0 s} \right) \left( 1 - \frac{1}{|\beta_{\lambda_0}|} \right) \right) N(\lambda_0, \delta_0; \phi) = h_{\lambda_0, \delta_0} N(\lambda_0, \delta_0; \phi), \\
M(\lambda_0, \delta_0; \phi) &\leq \max_{-T \leq t \leq 0} \left\{ e^{-(\lambda_0 + \delta_0)t} |\phi(t)| \right\} + \frac{|L(\lambda_0; \phi)|}{|\beta_{\lambda_0}|} \max_{-T \leq t \leq 0} \left\{ e^{-\lambda_0 t} \right\} + \frac{|R(\lambda_0, \delta_0; \phi)|}{\eta_{\lambda_0, \delta_0}} \\
&\leq \left( 1 + \frac{k_{\lambda_0} e_{\delta_0}}{|\beta_{\lambda_0}|} + \frac{h_{\lambda_0, \delta_0}}{\eta_{\lambda_0, \delta_0}} \right) N(\lambda_0, \delta_0; \phi).
\end{align*}
\]

Hence, from (37), we conclude that for all \( t \geq 0 \),

\[ e^{-(\lambda_0 + \delta_0)t} |\phi(t)| \leq \left( 1 + \frac{k_{\lambda_0} e_{\delta_0}}{|\beta_{\lambda_0}|} + \frac{h_{\lambda_0, \delta_0}}{\eta_{\lambda_0, \delta_0}} \right) N(\lambda_0, \delta_0; \phi) \mu_{\lambda_0, \delta_0}, \] (38)

and consequently, (32) holds.

The proof of Theorem 3 is completed. \( \square \)

**Proof of Corollary.** Define \( ||\phi|| \equiv \max_{-T \leq t \leq 0} |\phi(t)| \). It follows that \( ||\phi|| \leq N(\lambda_0, \delta_0; \phi) \).

Now, let us assume that \( \lambda_0 \leq 0 \) and \( \lambda_0 + \delta_0 \leq 0 \). From (32), it follows that

\[
|x(t)| \leq \left( 1 + \frac{k_{\lambda_0} e_{\delta_0}}{|\beta_{\lambda_0}|} + \left( 1 + \frac{\mu_{\lambda_0, \delta_0}}{\eta_{\lambda_0, \delta_0}} \right) \right) N(\lambda_0, \delta_0; \phi), \quad \text{for every } t \geq 0.
\]

Since \( \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} > 1 \), by taking into account the fact that

\[ \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} + \left( 1 + \frac{k_{\lambda_0} e_{\delta_0}}{|\beta_{\lambda_0}|} \right) \mu_{\lambda_0, \delta_0} + \left( 1 + \frac{\mu_{\lambda_0, \delta_0}}{\eta_{\lambda_0, \delta_0}} \right) h_{\lambda_0, \delta_0} \geq 1, \]

we have

\[
|x(t)| \leq \left( \frac{k_{\lambda_0}}{|\beta_{\lambda_0}|} + \left( 1 + \frac{k_{\lambda_0} e_{\delta_0}}{|\beta_{\lambda_0}|} \right) \mu_{\lambda_0, \delta_0} + \left( 1 + \frac{\mu_{\lambda_0, \delta_0}}{\eta_{\lambda_0, \delta_0}} \right) h_{\lambda_0, \delta_0} \right) N(\lambda_0, \delta_0; \phi), \quad \text{for every } t \in [-r, \infty),
\]

which means that the trivial solution of (1) is stable.
Next, if \( \lambda_0 < 0 \) and \( \lambda_0 + \delta_0 < 0 \), then (32) guarantees that
\[
\lim_{t \to -\infty} x(t) = 0
\]
and so the trivial solution of (1) is asymptotically stable.

Finally, if \( \delta_0 > 0, \lambda_0 + \delta_0 > 0 \), then the trivial solution of (1) is unstable. Otherwise, there exists a number \( \ell \equiv \ell(1) > 0 \) such that, for any \( \phi \in C^1(]-r, 0], IR) \) with \( ||\phi|| < \ell \), the solution \( x \) of problem (1) and (2) satisfies
\[
|x(t)| < 1 \quad \text{for all } t \geq -r. \quad (39)
\]

Define
\[
\phi_0(t) = e^{(\lambda_0 + \delta_0)t} - e^{\lambda_0 t} \quad \text{for } t \in [-r, 0].
\]
Furthermore, by the definition of \( L(\lambda_0; \phi) \) and \( R(\lambda_0, \delta_0; \phi) \), by using (5), we have
\[
L(\lambda_0; \phi_0) = \delta_0 + \int_{-r}^{0} f(s) e^{-\delta_0 s} \left( \int_{-r}^{s} e^{-\lambda_0 \tau} d\tau \right) ds - \int_{-r}^{0} f(s) e^{-\delta_0 s} ds = -2\lambda_0 - \int_{-r}^{0} f(s) e^{-\lambda_0 s} ds \equiv -\beta_{\lambda_0},
\]
\[
R(\lambda_0, \delta_0; \phi_0) = 1 - \int_{-r}^{0} f(s) e^{-\lambda_0 s} \left( \int_{-r}^{s} e^{-\delta_0 \tau} \left( \int_{-r}^{\tau} e^{-\lambda_0 \eta} \left( e^{(\lambda_0 + \delta_0)\eta} - e^{\lambda_0 \eta} \right) + 1 \right) d\eta \right) d\tau \left. \right|_{\tau = s} ds
\]
\[
= 1 - \delta_0 e^{-\delta_0 t} \left( 1 - e^{-\lambda_0 t} - \delta_0 e^{-\delta_0 t} \right) ds \equiv \eta_{\lambda_0, \delta_0} > 0.
\]

Let \( \phi \in C^1(]-r, 0], IR) \) be defined by
\[
\phi = \frac{\ell_1}{||\phi_0||} \phi_0,
\]
where \( \ell_1 \) is a number with \( 0 < \ell_1 < \ell \). Moreover, let \( x \) be the solution of (1) and (2). From Theorem 2 it follows that \( x \) satisfies
\[
\lim_{t \to -\infty} \left\{ e^{-(\lambda_0 + \delta_0)t} x(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} e^{-\delta_0 t} \right\} = \lim_{t \to -\infty} \left\{ e^{-(\lambda_0 + \delta_0)t} x(t) + \frac{\ell_1}{||\phi_0||} e^{-\delta_0 t} \right\} = \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}} = \frac{\ell_1}{||\phi_0||} > 0.
\]

But, we have \( ||\phi|| = \ell_1 < \ell \) and hence from (39) and conditions \( \delta_0 > 0, \lambda_0 + \delta_0 > 0 \) it follows that
\[
\lim_{t \to -\infty} \left\{ e^{-(\lambda_0 + \delta_0)t} x(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} e^{-\delta_0 t} \right\} = 0.
\]
This is a contradiction.

The proof of Corollary is completed. \( \square \)

3. Examples

Example 1. Consider
\[
x'(t) + 2x(t) = \int_{0}^{t} 3e^{-r} x(t - s) ds, \quad t \geq 0,
\]
\[
x(t) = \phi(t), \quad -1 \leq t \leq 0.
\]
where \( \phi(t) \) is an arbitrary continuously differentiable initial function on the interval \([-1, 0]\). In this example we apply the characteristic equations (3) and (5). That is, the characteristic equation (3) is
\[
\lambda^2 + 2 = \int_{0}^{1} 3e^{-s} e^{-\lambda s} ds,
\]
and we see that \( \lambda = -1 \) is a root of (41). Then, for \( \lambda_0 = -1 \) the characteristic equation (5) is
\[
\delta = 2 - \int_{0}^{1} 3e^{-s} e^{-\lambda s} \left( \int_{0}^{s} e^{-\delta \tau} d\tau \right) ds.
\]
Therefore, \( \delta = \delta_0 \equiv 0.84716 \) is a root, and the conditions of Corollary are satisfied. That is,
\[
\mu_{\lambda_0, \delta_0} \equiv 0.33326 < 1 \quad \text{and} \quad \beta_{\lambda_0} = \beta_{-1} = -\frac{1}{2} \neq 0.
\]
Since $\lambda_0 = -1 < 0$ and $\lambda_0 + \delta_0 = -0.15284 < 0$, the trivial solution of (40) is asymptotically stable.

**Example 2.** Consider
\[ x''(t) + x(t) = \int_0^1 x(t - s)ds, \quad t \geq 0, \]
\[ x(t) = \phi(t), \quad -1 \leq t \leq 0, \tag{42} \]
where $\phi(t)$ is an arbitrary continuously differentiable initial function on $[-1, 0]$. The characteristic equation (3) is
\[ \lambda^2 + 1 = \int_0^1 e^{-\lambda s}ds, \tag{43} \]
and we see easily that $\lambda = 0$ is a root of (43). Taking $\lambda_0 = 0$, the characteristic equation (5) is
\[ \delta = -\int_0^1 \left\{ \int_0^s e^{-\delta \tau}d\tau \right\}ds. \]
Therefore, we find that $\delta = \delta_0 \equiv -0.62203$ is a root. Corresponding to the roots $\lambda_0 = 0$ and $\delta_0 = -0.62203$, the conditions of Corollary are satisfied. Since $\lambda_0 = 0$ and $\lambda_0 + \delta_0 < 0$, the trivial solution of (42) is stable.

**Example 3.** Consider
\[ x''(t) - \frac{1}{2}x(t) = \int_0^1 e^{-s}x(t - s)ds, \quad t \geq 0, \]
\[ x(t) = \phi(t), \quad -\frac{1}{2} \leq t \leq 0, \tag{44} \]
where $\phi(t)$ is an arbitrary continuously differentiable initial function on $[-\frac{1}{2}, 0]$. The characteristic equation (3) is
\[ \lambda^2 - \frac{1}{2} = \int_0^1 e^{-s}e^{-\lambda s}ds, \tag{45} \]
and we see easily that $\lambda = -1$ is a root of (45). Taking $\lambda_0 = -1$, the characteristic equation (5) is
\[ \delta = 2 - \int_0^1 \left\{ \int_0^s e^{-\delta \tau}d\tau \right\}ds. \]
Therefore, we find that $\delta = \delta_0 \equiv 1.9068$ is a root. Corresponding to the roots $\lambda_0 = -1$ and $\delta_0 = 1.9068$, the conditions of Corollary are satisfied. Since $\delta_0 > 0$ and $\lambda_0 + \delta_0 > 0$, the trivial solution of (44) is unstable.

**References**