Backward Wave Modes of Partially Plasma Column Loaded Cylindrical Waveguide

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Abstract—When two waves traveling in the same or opposite direction are interfered, they can attenuate or increase each other. The backward wave oscillators work with the same principal that interference between two waves traveling in opposite directions increases the amplitude along propagation direction. There are backward wave oscillators which use interaction of plasma-electron beam, in the literature. The Method of Moment (MoM) is a widely used technique for numerical simulation of propagation and scattering problems. In this study, backward wave modes of the plasma column loaded cylindrical waveguide have been investigated by using Method of Moment (MoM). In previous studies, the authors presented the validation of the method for gyro-resonance region. For the structure, the backward waves appear in the plasma resonance region. Unlike gyro-resonance region modes (called plasma modes), to obtain the plasma resonance region modes (called cyclotron modes), it is necessary to use very large dimensions for the linear algebraic equations system used in the MoM. Indeed in the implementation of the MoM in the plasma resonance region for the cyclotron modes, we have found that the method exhibits discrepancies with the exact solution which do not disappear unless very large numbers of expansion functions are used. The objective of this work is to report these discrepancies and suggest a possible numerical technique to overcome them. Also presented are the dispersion curves of the backward wave modes. This technique consists in using additional computer time and memory space. In particular it is demonstrated that the mean relative error is confined to less than 0.001 over the whole frequency band of interest which would be impossible if insufficient computing power was used.

1. INTRODUCTION

In this study, backward wave modes of the plasma column loaded cylindrical waveguide have been investigated by using Method of Moment (MoM). A wave is considered a backward (forward) wave if its group velocity, as indicated by slope of dispersion curve, is opposite (the same) in direction to the phase velocity. The phase velocity \( v_{ph} \) and group velocity \( v_{gr} \) are described in (1).

\[
v_{ph} = \frac{\omega}{k}, \quad v_{gr} = \frac{d\omega}{dk}
\]

where, \( \omega \) is the operating frequency and \( jk \) is the wave propagation constant.

In this study, this definition, which is based on the dispersion properties of the wave propagation constant with respect to frequency, has been used for backward wave because the method used in the study directly gives the relation between the propagation constant and frequency. The variation of the electromagnetic fields is given below.

\[
F(r, \phi, z) = F(r) e^{j(kz+n\phi-\omega t)}
\]

where, \( n \) is the azimuthal variation number and \( r, \phi, z \) are the cylindrical coordinates. The tensor permittivity of the plasma column is equal to below matrix.

\[
\hat{\varepsilon} = \varepsilon_0 \begin{bmatrix} \varepsilon_1 & j\varepsilon_2 & 0 \\ -j\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}
\]

where, \( \varepsilon_0 \) is the permittivity of free space and the expressions of \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) in Equation (3) are given in (4). Their values change depending on the plasma frequency, the cyclotron frequency (so the material from which the plasma is produced) and the operating frequency.

\[
\varepsilon_1 = 1 + \frac{1}{R^2 - \Omega^2}, \quad \varepsilon_2 = \frac{-R}{\Omega(R^2 - \Omega^2)}, \quad \varepsilon_3 = 1 - \frac{1}{\Omega^2}
\]

where, \( R \) is the normalized cyclotron frequency (\( \omega_c/\omega_p \)) and \( \Omega \) is the normalized operating frequency (\( \omega/\omega_p \)). Besides, \( \omega_c \) is the cyclotron frequency and \( \omega_p \) is the plasma frequency. The backward waves in plasma column loaded cylindrical waveguide are called the cyclotron modes and appear in the...
plasma resonance region [1] where the normalized operating frequency is between max(1, $R$) and $\Omega_u$. Here, $\Omega_u$ is normalized upper hybrid frequency and described as below.

$$\Omega_u = \sqrt{1 + R^2}$$  \hfill (5)

The exact solution for the plasma column loaded cylindrical waveguide has been presented in different forms in the previous studies [1–5]. The exact solution of the structure has been used to check the accuracy of the solutions obtained from the MoM. The MoM is based on converting Maxwell’s partial differential equations into linear algebraic equations that are written in matrix form. In the presented study, “Generalized Telegraphist’s Equations” [6] or the transmission line equations have been utilized to generate linear algebraic equations system of the plasma column cylindrical waveguide and the validation of the method has been presented for the backward waves of the structure. Actually, the technique is a Fourier-Bessel series expansion technique and the number of expansion functions determines the dimension of the linear algebraic equations system.

In the previous studies, the authors presented the validation of the method for gyro-resonance region [7, 8]. Besides, they compared the method with two semi analytical methods, the quasi-static approximation and the asymptotic approximation, and they showed that the MoM is a better method than the other two methods for all frequencies of the gyro-resonance region [9]. Unlike gyro-resonance region modes (called plasma modes), to obtain the cyclotron modes, it is necessary to use very large dimensions for the linear algebraic equations system. Indeed in the implementation of the MoM in the plasma resonance region for the cyclotron modes, it has been observed that the method exhibits discrepancies with the exact solution which do not disappear unless very large numbers of expansion functions are used. In particular it is demonstrated that the mean relative error which equals to average of the relative error for whole frequency interval is confined to less than 0.001 over the whole frequency band of interest which would be impossible if insufficient computing power was used. Hence a numerical approach to solve this problem has been observed to be increasing the number of expansion functions.

In the study, the second section explains the MoM. In the third section, obtained results have been given. The last section constitutes the conclusion.

2. THE METHOD OF MOMENT

Maxwell’s equations do not have an exact solution in the closed form for every physical structure. When the exact solution does not exist for any physical structure, a numerical solution, as the finite difference method or the finite element method, or a semi analytical method, as the transmission line method or the MoM, is investigated in order to obtain the solution. One of the best known semi-analytical methods for closed and sourceless waveguides was given by Schelkunoff in 1952 [6]. In his classical study, Schelkunoff derived the transverse field component from the potential and the stream functions for general structure of closed waveguides. The method transforms Maxwell’s equations, consisting of partial differential equations, into an ordinary differential equations system containing differentiation with respect to propagation direction ($z$). The obtained system is also called the transmission line model of the structure. If Equation (2) is considered, the differentiation with respect to propagation direction is equivalent to multiplication with $jk$. Thus, the ordinary differential equations system is transformed into a linear algebraic equations system. The linear algebraic equations system for fully/partially gyro-electric or gyro-magnetic medium loaded waveguide is in the form of Equation (6).

$$jk(p) \begin{bmatrix} v(p) \\ i(p) \end{bmatrix} = \begin{bmatrix} 0 & Z(p) \\ Y(p) & 0 \end{bmatrix} \begin{bmatrix} v(p) \\ i(p) \end{bmatrix}$$ \hfill (6)

In this way, the problem of electromagnetic propagation in the gyro-electric medium loaded cylindrical waveguide is converted into an eigenvalue problem. In expression (6), $p$ is the complex frequency, $jk(p)$ shows complex propagation constant, $v(p)$ and $i(p)$ are the unknown voltage and the current vectors, respectively. Besides, $Z(p)$ and $Y(p)$ are the complex impedance and admittance coefficient matrices per unit length, respectively.

The eigenvalues of the impedance-admittance coefficient matrix in (6) gives the propagation constants of the problem. The dimension of the system is determined from the number of known solutions of the empty waveguide used in the Fourier-Bessel expansion. The method is called as a semi-analytical method because of necessity of truncating an infinite summation of series at a point, while it uses known analytic solutions of the empty waveguide. This method is also called Galerkin version of the MoM [10].
3. THE BACKWARD WAVE MODES

The modes existing in the plasma resonance region are the backward wave modes. These modes have been presented as cyclotron modes [1], dynamic modes [2] and HE modes [3]. In the study, the plasma-resonance region’s modes are called cyclotron modes as in [1] and symbolized with $R_{\delta}C_{n,m}$. Here, $\delta$ is the normalized waveguide radius ($\omega_p a/c$) and $m$ is the mode number. Also, $a$ and $c$ are the radius of the waveguide and the velocity of light in vacuum, respectively.

In the previous studies, the authors presented the plasma modes existing in gyro-resonance region by using the MoM. Unlike plasma modes, to obtain the cyclotron modes, it is necessary to use very large dimensions for the linear algebraic equations system used in the MoM. As a consequence of numerical computations for cyclotron modes, it is observed that the method exhibits discrepancies with the exact solution which do not disappear unless very large numbers of expansion functions ($N$) are used. These discrepancies occur at different frequency points for different number of expansion functions and disappear while the term number of expansion functions enlarges as shown Figure 1. At frequency points other than the discrepancy points, shown within ellipses in

- N=50
- N=250
- N=1000

Figure 1: The dispersion curves obtained from the MoM for different number of expansion functions.

- R = 0.5, s_0 = 0.9, $\delta = 1.0$

R = 0.5, s_0 = 0.9, $\delta = 1.0$

Figure 2: The dispersion curves obtained from the exact solution and the MoM for $R = 0.5$ and $s_0 = 0.9$. 

- N=50
- N=250
- N=1000

Figure 1: The dispersion curves obtained from the MoM for different number of expansion functions.
In the figure, the values obtained from the method conform with the exact solutions. In the figures, $\gamma$ stands for $k/k_0$ where $k_0$ is the wave number in free space.

The dimension of the coefficients matrix is taken as $2N \times 2N$. In order to overcome the discrepancies, it is necessary to enlarge number of expansion functions (or augment dimension of the coefficient matrix), which means using additional computer time and memory space.

In numeral computations, first the number of expansion functions, which confines the mean relative error to less than 0.001 over the whole frequency band, has been determined for each structure. The numerical computations have been performed by using MATLAB R2009b on a computer which has Intel® Core™ i7 CPU 960 @ 3.20 GHz, 12 GB RAM and 64 bit operating system. As a consequence of computations, it is found that it is necessary to use bigger number of expansion functions in order to eliminate the discrepancies while the plasma column radius decreases in waveguide. The plasma to waveguide ratio is described as $s_0 = b/a$. Here $b$ denotes the radius of the plasma column.

In this study, the first three degrees of the cyclotron modes ($m = 1, 2, 3$) have been investigated.

Figure 3: The dispersion curves obtained from the exact solution and the MoM for $R = 1.5$ and $s_0 = 0.5$.

Figure 4: The dispersion curves obtained from the exact solution and the MoM for $R = 0.5$ and $s_0 = 0.1$. 
using the MoM and compared with the exact solution given in [1]. For numerical computation, the waveguide radius, the normalized waveguide radius, the azimuthal variation and the plasma frequency have been taken fixed as $a = 3$ cm, $\delta = 1$, $n = 1$ and $\omega_p = 10^{10}$ rad/s, respectively. The numerical computations have been performed for two groups of the normalized cyclotron frequency. The first is for relatively weak magnetic field ($R = 0.5$) and the second is for the relatively strong magnetic field ($R = 1.5$) as described in [1]. The investigated structures have three different ratios of radii ($s_0 = 0.1, 0.5, 0.9$). As the result, the numbers of expansion functions which confine mean relative error to less than 0.001 have been determined as $N = 3000$ for $s_0 = 0.9$, $N = 4000$ for $s_0 = 0.5$ and $N = 5000$ for $s_0 = 0.1$ for both groups of the normalized cyclotron frequencies. Dispersion curves are given for all three ratios of radii while $R = 0.5$ or $R = 1.5$. The $R = 0.5$ and $s_0 = 0.9$ case, $R = 1.5$ and $s_0 = 0.5$ case, $R = 0.5$ and $s_0 = 0.1$ case are depicted respectively in Figures 2, 3, 4. The results obtained from the MoM conform with the exact solutions pretty well. In order to show differences of the results, the dispersion curves have been plotted by zooming at some frequency regions in the figures.

4. CONCLUSION
It is to be noted that for smaller dimensions of the coefficients matrix in the MoM the spike like formations observed in Figure 1 do not disappear. Hence one needs as long CPU times as several hours for each run for each frequency in the band $1 < \Omega < 1.12$ for $R = 0.5$ and the band $1.5 < \Omega < 1.8$ for $R = 1.5$. This problem has been encountered only for the frequency intervals above the gyro-resonance frequency region. Increasing the dimension of the coefficient matrix has eliminated these discrepancies in the structures reported above and the necessary minimum dimensions have been listed for each structure. Lastly dispersion curves of the backward wave modes have been presented comparatively with the exact solutions.

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