Monotonicity of input–output mappings in inverse coefficient and source problems for parabolic equations

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Received 3 October 2006
Available online 13 February 2007
Submitted by B. Straughan

Abstract

This article presents a mathematical analysis of input–output mappings in inverse coefficient and source problems for the linear parabolic equation \( u_t = (k(x)u_x)_x + F(x,t), \quad (x,t) \in \Omega_T := (0,1) \times (0,T) \). The most experimentally feasible boundary measured data, the Neumann output (flux) data \( f(t) := -k(0)u_x(0,t) \), is used at the boundary \( x = 0 \). For each inverse problems structure of the input–output mappings is analyzed based on maximum principle and corresponding adjoint problems. Derived integral identities between the solutions of forward problems and corresponding adjoint problems, permit one to prove the monotonicity and invertibility of the input–output mappings. Some numerical applications are presented.

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Keywords: Inverse coefficient and source problems; Parabolic equation; Monotonicity of input–output mappings; Adjoint problems

1. Introduction

Consider the following initial boundary value problems:

\( u_t = (k(x)u_x)_x + F(x,t), \quad (x,t) \in \Omega_T := (0,1) \times (0,T) \)
\( u(x,0) = g(x), \quad x \in (0,1) \)
\( u(0,t) = f(t), \quad t \in (0,T) \)
\( u(1,t) = h(t), \quad t \in (0,T) \)

This work was partially supported by the Scientific and Technical Research Council of Turkey (TUBITAK). The research of the first author was also supported by the Al-Farabi Kazakh National University.

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0022-247X/$ – see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2007.01.097
\[
\begin{aligned}
&u_t(x,t) = \left(k(x)u_x(x,t)\right)_x, \quad (x,t) \in \Omega_T, \\
u(x,0) = 0, \quad 0 < x < 1, \\
u(0,t) = g(t), \quad k(1)u_x(1,t) = 0, \quad 0 < t < T,
\end{aligned}
\]
where \( \Omega_T = \{(x,t) \in R^2: 0 < x < 1, \quad 0 < t \leq T\} \). The functions \( k(x) > 0 \) and \( g(t) \geq 0 \) satisfy the following conditions:

\begin{enumerate}
\item[(C1)] \( k(x) \in C^1[0, 1], \ c_1 > k(x) > c_0 > 0; \)
\item[(C2)] \( g(t) \in C[0, T]. \)
\end{enumerate}

Under these conditions the initial boundary value problem (1) has the unique solution \( u(x,t) \in C^2_0(\Omega_T) \cap C_{1,0}(\overline{\Omega_T}) \) [14].

Consider the following inverse problem of determining the unknown coefficient \( k = k(x) \) from the flux data \( f(t) \) at the boundary \( x = 0 \), defined by

\[
f(t) := -k(0)u_x(0,t), \quad t \in (0, T].
\]

The function \( f(t) \) is defined to be the Neumann type of measured output data. Let us denote by \( K \subset C^1[0, 1] \) the set of admissible coefficients \( k = k(x) \), and by \( u(x, t; k) \) the unique solution of problem (1), corresponding to this coefficient. Then the function

\[
\tilde{f}(t; k) := -k(0)u_x(0,t; k), \quad t \in (0, T],
\]
will be defined to be the Neumann output data. We denote by \( f \subset C[0, T] \) the set of measured output data \( f(t) \). Then the inverse problem (1)–(2) can be formulated in the following operator form:

\[
\Phi[k](t) = f(t), \quad t \in (0, T].
\]

According to [7–9], the mapping \( \Phi[\cdot]: K \rightarrow f, \Phi[k](\cdot) := -k(0)u_x(0, \cdot; k) \), is defined to be the input–output or coefficient-to-data mapping.

Therefore the inverse problem (1)–(2) with the Neumann measured output data \( \tilde{f}(t) \) can be reduced to the solution of the nonlinear equation (3) or to the problem of inverting the input–output map \( \Phi: K \mapsto f \).

The problem of identifying the unknown coefficient \( k(x) \) from the boundary measured or final state data is a very important inverse problem in many areas, including heat conduction, diffusion [1,3], oil reservoir simulation and groundwater flow [2] (see also [11] and references therein). These problems are known to be severely ill-posed, i.e. the small perturbations in the boundary measured data cause a dramatically large error in the solution. The methods related to an existence of a solution of such inverse problems can be separated into two general groups:

I. Output Least Squares (OLS), based on the notion of quasisolution given in [12,14].

II. Monotonicity methods, based on integral relationships between the input and output data.

The first group methods are widely used one (see [1,4,6,12–14] and references therein). Here the measured output data is used to define the error functional \( J(k) := \|\Phi[k] - f\|^2 \) by using an appropriate norm \( \| \cdot \| \), and a quasisolution of the inverse problem (1)–(2) is defined as a solution of a minimization problem for the functional \( J(k) \) over the set of admissible coefficients \( K \). The relationship between the inputs and outputs in these methods can only be expressed indirectly, through the solver. Hence general information about properties of input–output mapping \( \Phi[\cdot]: K \rightarrow f \) is not readily available by OLS methods.
Monotonicity methods [7–9] permit one to construct an integral relationship between the input and output data, which contains the solution of corresponding adjoint problem. This integral relationship with maximum principle allows to describe the structure of the input–output mapping, in particular, its monotonicity. Specifically, this approach clearly display the connection between the input and output by an invertible mapping.

In this paper we give a systematic analysis of input–output mappings for two widely used inverse problems. The first inverse problem we consider is above problem of determining the unknown coefficient $k(x)$ from the Neumann measured data $f(t)$. As a second inverse problem we consider the problem of identification the unknown source term $F(x,t)$ in the parabolic equation $u_t(x,t) = (k(x)u_x(x,t))_x + F(x,t)$ from the same Neumann data $f(t)$, defined by (2). When the both boundary conditions in the direct problem is of Neumann type, the monotonicity of the input–output mapping for the first inverse problem is derived in [10]. However the case of mixed boundary conditions, given in (1), the corresponding adjoint problem, as well as an integral relationship between the direct and adjoint problem solutions, are different. For the both considered inverse problems we obtain integral relationships between the solutions of the direct and corresponding adjoint problems, which contain also output data $\tilde{f}(t; k)$. Choosing arbitrary (control) functions in these adjoint problems we prove monotonicity, Lipschitz continuity, and hence the invertibility of input–output mappings.

The rest of this paper is organized as follows. In Section 2, we first prove that the values $k(0)$ and $k(1)$ of the unknown coefficient $k(x)$ at the endpoints $x = 0$ and $x = 1$ can be found explicitly, via the Dirichlet data $g(t)$ of the direct problem (1) and the corresponding Green function. This result permits one to use the values $k(0)$ and $k(1)$ in subsequent numerical method. Note that similar formula obtained in [10] for this value, contains the output data $f(t)$ which can be given with some noise. Then we deduce some properties of the solution of the direct problem (1) from the properties of the input data $g(t)$. Monotonicity and invertibility of the input–output mapping $\Phi[\cdot]: \mathcal{K} \to \mathbf{f}$ for the inverse problem (1)–(2) is discussed in Section 3. An analysis of the considered approach for the inverse source problem with single Neumann data $f(t)$ is given in Section 4. In Section 5 we illustrate some numerical examples to show usefulness of obtained integral identities.

2. Some properties of the direct problem solution

First we establish an analytical formula for the values $k(0)$ and $k(1)$ of the unknown coefficient $k(x)$ at the endpoints $x = 0$ and $x = 1$ of the considered interval $[0, 1]$, via data $g(t)$ of the direct problem and the corresponding Green function.

**Lemma 1.** Let $g(t) > 0$ for all $t \in (0, T]$ be a given input data in the direct problem (1). Then the values $k(0)$ and $k(1)$ of the unknown diffusion coefficient $k = k(x)$ can be determined from this data as follows

$$
k(0) = \lim_{t \to 0} \frac{g(t)}{G_0(t)}, \quad k(1) = \lim_{t \to 0} \frac{g(t)}{G_1(t)},$$

where
\[\begin{align*}
\hat{G}_0(t) &= -2 \int_0^t \frac{\partial \Theta_0(0, t - \tau)}{\partial x} g(\tau) d\tau, \\
\hat{G}_1(t) &= 2 \int_0^t \frac{\partial \Theta_1(1, t - \tau)}{\partial x} g(\tau) d\tau,
\end{align*}\] (5)

and \(G_0(x,t)\) and \(G_1(x,t)\) are the Green functions for the parabolic equations 
\[v_i^{(0)} = k(0)v_{xx}^{(0)}, \quad v_i^{(1)} = k(1)v_{xx}^{(1)},\]
respectively.

**Proof.** Let us define the function \(v^{(0)}(x,t) = u(x,t; k(0))\). Then \(v^{(0)}(x,t)\) is the solution of the following problem:

\[\begin{align*}
v_t^{(0)}(x,t) - k(0)v_{xx}^{(0)}(x,t) &= 0, \quad (x,t) \in \Omega_T, \\
v^{(0)}(x,0) &= 0, \quad x \in (0, 1), \\
v^{(0)}(0,t) &= g(t), \quad v^{(0)}_x(1,t) = 0, \quad t \in (0,T).
\end{align*}\] (6)

The solution \(v^{(0)} = v^{(0)}(x,t)\) of problem (6) is defined as follows [5]

\[v^{(0)}(x,t) = -2k(0) \int_0^t \frac{\partial G_0(x,t - \tau)}{\partial x} g(\tau) d\tau, \quad (x,t) \in \Omega_T.\] (7)

We define now the function \(w(x,t) = u(x,t) - v^{(0)}(x,t)\). Then we have

\[w_t = u_t - v_t^{(0)} = (k(x)u_x)_x - k(0)v_{xx}^{(0)} = ((k(x) - k(0))u_x)_x + k(0)(u - v^{(0)})_{xx}.\]

Hence the function \(w = w(x,t)\) is the solution of the following initial boundary value problem:

\[\begin{align*}
w_t - k(0)w_{xx} &= F_1(x,t), \quad (x,t) \in \Omega_T, \\
w(x,0) &= 0, \quad w(0,t) = w_x(1,t) = 0,
\end{align*}\]

where \(F_1(x,t) = ((k(x) - k(0))u_x)_x\). For an arbitrary test function \(\psi(x,t)\), we have

\[\int_{\Omega_T} \psi(x,t)[w_t - k(0)w_{xx}] dx dt = \int_{\Omega_T} \psi(x,t)((k(x) - k(0))u_x)_x dx dt.\]

We require that \(\psi(x,t)\) solves the adjoint problem

\[\begin{align*}
v_t^{(0)}(x,t) + k(0)v^{(0)}_x(x,t) &= F(x,t), \quad (x,t) \in \Omega_T, \\
v^{(0)}(x,T) &= 0, \quad v^{(0)}(0,t) = v^{(0)}_x(1,t) = 0,
\end{align*}\]

where \(F(x,t) \in C(\Omega_T)\) is an arbitrary function. Then it follows from integration by parts that

\[\int_{\Omega_T} w(x,t)F(x,t) dx dt = \int_{\Omega_T} \int (k(x) - k(0))u_x \psi_x dx dt.\] (8)

Since \(k(x)\) is continuous, for each \(\varepsilon > 0\) there is \(\delta > 0\) such that \(|k(x) - k(0)| < \varepsilon\) if \(0 < x < \delta\). Then
\[ \left| \int_0^T \int_{\Omega_T} (k(x) - k(0)) u_x \psi_x \, dx \, dt \right| \leq \left| \int_0^\delta (k(x) - k(0)) \int_0^T u_x \psi_x \, dx \, dt \right| \\
+ \left| \int_\delta^1 (k(x) - k(0)) \int_0^T u_x \psi_x \, dx \, dt \right| \\
\leq \varepsilon C_1 + c_k \left| \int_0^\delta \int_0^T u_x \psi_x \, dx \, dt \right| , \]

where
\[ C_1 = \left| \int_0^\delta \int_0^T u_x \psi_x \, dx \, dt \right| \quad \text{and} \quad c_k = \max_{[\delta, 1]} |k(x) - k(0)|. \]

The right-hand side \( F(x, t) \) in the adjoint problem for \( \psi(x, t) \) is an arbitrary function and we may require that \( \text{Supp} \ F(x, t) \subset (0, \eta) \times (0, \tau) \) and \( F(x, t) \equiv 1, \ (x, t) \in \text{Supp} \ F(x, t) \). Then due to the continuity of the adjoint problem solution \( \psi(x, t) \) with respect to \( F(x, t) \), we can choose \( \eta > 0 \) and \( \tau > 0 \) such that
\[ \left| \int_0^\delta \int_0^T u_x \psi_x \, dx \, dt \right| \leq \varepsilon. \]

Thus
\[ \left| \int_0^\delta \int_0^T (k(x) - k(0)) u_x \psi_x \, dx \, dt \right| \leq C_3 \varepsilon, \quad C_3 = C_1 + c_k. \]

The point we try to show here is that as \( \eta \) and \( \tau \) are getting smaller, \( w(x, t) = u(x, t; k(x)) - v(x, t) \) is getting smaller as well. This follows from the above estimate and the integral equation (8):
\[ \left| \int_\Omega_T w(x, t) \, dx \, dt \right| \leq C_3 \varepsilon, \quad \forall \varepsilon > 0. \]

Hence
\[ \lim_{t \to 0} \lim_{x \to 0} w(x, t) = 0, \quad \text{i.e.} \quad \lim_{t \to 0} \lim_{x \to 0} \left( u(x, t; k(x)) - v(0)(x, t) \right) = 0, \]
or equivalently
\[ \lim_{x \to 0} \lim_{t \to 0} \left[ \frac{u(x, t; k(x))}{v(0)(x, t)} \right] = 1. \]

By (5) and (7) we get
\[ \lim_{t \to 0} \frac{u(0, t; k(x))}{u(0, t; k(0))} = \lim_{t \to 0} \frac{g(t)}{k(0) \widehat{G_0(t)}} = 1, \]

which implies (4).
The second can be obtained by a similar way, introducing the function $v^{(1)}(x,t)$. In this case instead of problem (6) we need to take the problem

$$\begin{cases}
    v^{(1)}_t(x,t) - k^{(1)}v^{(1)}_{xx}(x,t) = 0, & (x,t) \in \Omega_T, \\
    v^{(1)}(x,0) = 0, & x \in (0,1), \\
    v^{(1)}(0,t) = g(t), & v^{(1)}_x(1,t) = 0, & t \in (0,T),
\end{cases}$$

with the solution

$$v^{(1)}(x,t) = 2k^{(1)} \int_0^t \frac{\partial G_1(x,t - \tau)}{\partial x} g(\tau) \, d\tau, \quad (x,t) \in \Omega_T.$$ 

The lemma is proved. \hfill \square

**Corollary 1.** Let $u_1(x,t) = u(x,t; k_1)$ and $u_2(x,t) = u(x,t; k_2)$ be two solutions of the direct problem (1) corresponding to the admissible coefficients $k_1(x), k_2(x) \in \mathcal{K}$. Then these coefficients satisfy the conditions \( k_1(0) = k_2(0), k_1(1) = k_2(1). \)

The result below shows an influence of the sign of the input Dirichlet data $g = g(t)$ to the sign of the output data $\tilde{f}(t;k)$.

**Theorem 1.** Let $u = u(x,t)$ be the solution of problem (1) and conditions (C1)--(C2) hold. Assume, in addition, that $u_x(x,t)$ is continuous on the closure of $\Omega_T$. If $g(t) > 0$ for $0 < t < T$, then $u_x(x,t) < 0$, a.e. $\forall (x,t) \in \Omega_T$.

**Proof.** Let $\varphi(x,t) \in \mathcal{D} := C_0^\infty(\mathbb{R}^2)$ be an arbitrary smooth function with compact support $\mathcal{D}$ in $\Omega_T$. Multiply the both sides of Eq. (1) by $\varphi_t(x,t)$:

$$\int_{\Omega_T} \int [u_t - (k(x)u_x)_x] \varphi_x \, dx \, dt = 0.$$ 

Integration by parts yields

$$\int_0^1 (u \varphi_x)^{t=T}_{t=0} \, dx - \int_{\Omega_T} u \varphi_{xt} \, dx \, dt - \int_0^T (k(x)u_x \varphi_x)|_{x=0}^{x=1} \, dt + \int_{\Omega_T} k(x)u_x \varphi_{xx} \, dx \, dt = 0.$$ 

Again we apply integration by parts to the second integral:

$$\int_0^1 (u \varphi_x)^{t=T}_{t=0} \, dx - \int_0^T (u \varphi_t)|_{x=0}^{x=1} \, dt + \int_{\Omega_T} u_x \varphi_t \, dx \, dt$$

$$- \int_0^T (k(x)u_x \varphi_x)|_{x=0}^{x=1} \, dt + \int_{\Omega_T} k(x)u_x \varphi_{xx} \, dx \, dt = 0.$$ 

Hence
\[
\int_{\Omega_T} \int_T u_x(\varphi_t + k(x)\varphi_{xx}) \, dx \, dt = \int_0^T (u\varphi_t)|_{x=0}^1 \, dt + \int_0^T (k(x)u_x\varphi_x)|_{x=0}^1 \, dt - \int_0^1 (u\varphi_x)|_{t=0}^T \, dx. \quad (9)
\]

Now we require that the function \( \varphi(x, t) \) is chosen to be the solution of the following backward parabolic equation:
\[
\varphi_t + k(x)\varphi_{xx} = F(x, t), \quad (x, t) \in \Omega_T,
\]
where an arbitrary continuous function \( F(x, t) \) will be defined below. Since \( \text{Supp} \varphi(x, t) \subset \Omega_T \), the function \( \varphi(x, t) \) also satisfies the following homogeneous boundary and final \((t = T)\) conditions:
\[
\left\{ \begin{array}{l}
\varphi(0, t) = \varphi(1, t) = 0, \quad t \in (0, T), \\
\varphi(x, T) = 0, \quad x \in (0, 1).
\end{array} \right. \quad (11)
\]
Note that Eq. (10) with boundary and the final conditions (11) constitutes a backward initial boundary value problem for the function \( \varphi(0, t) \). This problem is well-posed because if \( t \in [0, T] \) is replaced by \( \tau = -t \) in Eq. (10), the parabolic equation \( \varphi_\tau = k(x)\varphi_{xx} - F(x, -\tau) \) will be obtained. Hence the final boundary value problem (10)–(11) is completely specified.

The final and boundary conditions (11) imply \( \varphi_t(0, t) = \varphi_t(1, t) = \varphi_x(x, T) = 0 \). Substituting these on the right-hand side of (9) and taking into account the backward equation (10) and the condition \( u_x(1, t) = 0 \), we get
\[
\int_{\Omega_T} \int_T u_x(x, t)F(x, t) \, dx \, dt = -\int_0^T k(0)u_x(0, t)\varphi_x(0, t) \, dt. \quad (12)
\]
Now we apply the maximum principle to the adjoint problem (10)–(11). We require that the function \( F(x, t) \) satisfies the condition \( F(x, t) > 0 \) on \( \Omega_T \). Thus \( \varphi(x, t) < 0 \) on \( \Omega_T \). This, with the boundary condition \( \varphi(0, t) = 0 \), implies
\[
\varphi_x(0, t) := \lim_{h \to 0} \frac{\varphi(h, t) - \varphi(0, t)}{h} < 0.
\]
On the other hand, taking into account \( g(t) = u(0, t) > 0 \) and applying the maximum principle, we get \( 0 < u(x, t) < g(t) \). This implies
\[
u_x(0, t) := \lim_{h \to 0} \frac{u(h, t) - u(0, t)}{h} < 0.
\]
Hence the right-hand side of (12) is negative, i.e.,
\[
\int_{\Omega_T} \int_T u_x(x, t)F(x, t) \, dx \, dt < 0, \quad \forall F(x, t) > 0.
\]
This implies \( u_x(x, t) < 0 \), for all \((x, t)\) in \( \Omega_T \). □

The assumption \( g(t) > 0 \) in the problem (1), physically means heating at the left boundary \( x = 0 \) of the rod. By the assertion of the above theorem the flux \( \tilde{f}(t; k) := k(0)u_x(0, t) \) is positive. This result is compatible with the physical meaning of heat conduction.
Theorem 2. Let \( u = u(x, t) \) be the solution of problem (1) and conditions (C1)–(C2) hold. Assume, in addition, that the solution \( u(x, t) \) is continuously differentiable on the closure of \( \Omega_T \). If \( g'(t) > 0 \) for \( 0 < t < T \), then \( u_t(x, t) > 0, \forall (x, t) \in \Omega_T \).

Proof. Multiplying Eq. (1) by an arbitrary function \( \varphi(x, t) \in D \) with compact support in \( \Omega_T \), we get

\[
0 = \int \int_{\Omega_T} \left[ u_t - (k(x)u_x)_x \right] \varphi_t \, dx \, dt = \int \int_{\Omega_T} u_t \varphi_t \, dx \, dt - \int \int_{\Omega_T} (k(x)u_x)_x \varphi_t \, dx \, dt. \tag{13}
\]

Applying integration by parts to the last integral, we have

\[
- \int \int_{\Omega_T} (k(x)u_x)_x \varphi_t \, dx \, dt = - \int_0^1 \left. \left((k(x)u_x)_x \varphi \right) \right|_{t=0}^T \, dx + \int \int_{\Omega_T} (k(x)u_x)_x \varphi \, dx \, dt
\]

\[
= - \int_0^1 \left. \left((k(x)u_x)_x \varphi \right) \right|_{t=0}^T \, dx + \int_0^T \left. \left((k(x)u_x)_x \varphi \right) \right|_{x=0}^{x=1} \, dt - \int \int_{\Omega_T} (k(x)u_x)_x \varphi \, dx \, dt.
\]

We apply again integration by parts in the last integral. Then we have

\[
- \int \int_{\Omega_T} (k(x)u_x)_x \varphi_t \, dx \, dt = - \int_0^1 \left. \left((k(x)u_x)_x \varphi \right) \right|_{t=0}^T \, dx + \int_0^T \left. \left((k(x)u_x)_x \varphi \right) \right|_{x=0}^{x=1} \, dt
\]

\[
- \int_0^T \left. \left((k(x)u_x)_x \varphi \right) \right|_{x=0}^{x=1} \, dt + \int \int_{\Omega_T} (k(x)u_x)_x \varphi \, dx \, dt.
\]

This, with (13), implies

\[
\int \int_{\Omega_T} (\varphi_t + (k(x)\varphi_x)_x) u_t \, dx \, dt = \int_0^1 \left. \left((k(x)u_x)_x \varphi \right) \right|_{t=0}^T \, dx - \int_0^T \left. \left((k(x)u_x)_x \varphi \right) \right|_{x=0}^{x=1} \, dt
\]

\[
+ \int_0^T \left. \left((k(x)u_x)_x \varphi \right) \right|_{x=0}^{x=1} \, dt. \tag{14}
\]

Let \( \varphi = \varphi(x, t) \) be a solution of the following adjoint problem:

\[
\begin{aligned}
\varphi_t + (k(x)\varphi_x)_x &= F(x, t), \quad (x, t) \in \Omega_T, \\
\varphi(x, T) &= 0, \quad x \in (0, 1), \\
\varphi(0, t) &= \varphi_x(1, t) = 0, \quad t \in (0, T),
\end{aligned} \tag{15}
\]

where \( F(x, t) \) is an arbitrary continuous function. Then taking into account the homogeneous initial and boundary conditions (1) and (15) in (14) we get
\[ \int_\Omega T F(x,t)u_t(x,t) \, dx \, dt = -\int_0^T k(0)g'(t)\varphi(x,0,t) \, dt. \] (16)

Again requiring \( F(x,t) > 0, \forall (x,t) \in \Omega T \), implies \( \varphi(x,t) < 0 \). This, with \( \varphi_x(0,0,t) < 0 \) on \( \Omega T \), implies that the right-hand side of (16) is positive, i.e.,
\[ \int_\Omega T F(x,t)u_t(x,t) \, dx \, dt > 0, \quad \forall F(x,t) > 0, \]
and we have the proof. \( \square \)

3. Monotonicity and invertibility of the input–output mapping \( \Phi[\cdot] : K \to f \) in the inverse problem (1)–(2)

Let \( u_1(x,t) := u(x,t; k_1) \) and \( u_2(x,t) := u(x,t; k_2) \) be two solutions of direct problem (1) corresponding to the admissible coefficients \( k_1(x), k_2(x) \in K \). Denote by \( \bar{f}(t; k_j) = -k(0)u_x(0,t; k_j), j = 1, 2, \) is the corresponding outputs, and let \( \Delta \bar{f}(t) = \bar{f}(t; k_1) - \bar{f}(t; k_2), \Delta k(x) = k_1(x) - k_2(x) \).

Lemma 2. Let \( u_1(x,t) \) and \( u_2(x,t) \) be two solutions of direct problem (1) corresponding to the admissible coefficients \( k_1(x), k_2(x) \in K \). Then for each \( \tau \in (0,T] \) the output \( \bar{f}(t; k_j) \) satisfies the following integral identity:
\[ \int_0^\tau p(t) \Delta \bar{f}(t) \, dt = \int_\Omega T \Delta k(x)(u_2)_x(x,t)\varphi_x(x,t) \, dx \, dt, \] (17)
where the function \( \varphi(x,t) = \varphi(x,t; p) \) is the solution of the following adjoint problem
\[ \begin{cases} 
\varphi_t + (k_1(x)\varphi_x)_x = 0, & (x,t) \in (0,1) \times (0, \tau), \\
\varphi(x,\tau) = 0, & x \in (0,1), \\
\varphi(0,t) = p(t), & \varphi_x(1,t) = 0, & t \in (0, \tau),
\end{cases} \] (18)
with arbitrary Dirichlet data \( p(t) \in C(0,T] \).

Proof. Let \( w(x,t) = u_1(x,t) - u_2(x,t) \). Then by Eq. (1), \( w_t = (u_1)_t - (u_2)_t = (k_1(u_1)_x)_x - (k_2(u_2)_x)_x = (k_1(u_1 - u_2)_x)_x + ((k_1 - k_2)(u_2)_x)_x \). Hence \( w = w(x,t) \) solves the following initial boundary value problem:
\[ \begin{cases} 
w_t - (k_1 w_x)_x = \Delta k(u_2)_x(x,t), & (x,t) \in (0,1) \times (0, \tau), \quad 0 < \tau \leq T, \\
w(x,0) = 0, & x \in (0,1), \\
w(0,t) = 0, & -k_1(1)w_x(1,t) = \Delta k(1)(u_2)_x(1,t), & t \in (0, \tau).
\end{cases} \]

Multiplying each side of the above equation by an arbitrary function \( \varphi(x,t) \) and integrating by parts on \( \Omega T = (0,1) \times (0, \tau), 0 < \tau \leq T, \) we get
\[ -\int_\Omega T \Delta k(x)(u_2)_x(x,t)\varphi_x(x,t) \, dx \, dt + \int_0^\tau (\Delta k(u_2)_x \varphi)_{x=1}^{x=0} \, dt \]
\[ \int_0^1 w \varphi |^0 \, dx - \int_0^\tau (\varphi k_1 w_x - w k_1 \varphi_x) |^\tau = 0 \, dt - \int \int_{\Omega_T} w (\varphi_t + (k_1 \varphi)_x) \, dx \, dt. \] (19)

Now we require that the function \( \varphi = \varphi(x, t) \) is the solution of the adjoint problem (18). Then due to the homogeneous boundary conditions \( w(x, 0) = w(0, t) = 0 \), the integral identity (19) implies:

\[ - \int \int_{\Omega_T} \Delta k(x) (u_2)_x(x, t) \varphi_x(x, t) \, dx \, dt + \int_0^\tau \Delta k(1)(u_2)_x(1, t) \varphi(1, t) \, dt \]

\[ = \int_0^\tau (\Delta k(1)(u_2)_x(1, t) \varphi(1, t) + k_1(0) w_x(0, t) \varphi(0, t)) \, dt. \]

Taking into account

\[ - k_1(0) w_x(0, t) = - k_1(0)(u_1)_x(0, t) + k_1(0)(u_2)_x(1, t) = \tilde{f}_1(t) - \tilde{f}_2(t) = \Delta \tilde{f}(t), \]

we obtain the required integral identity (17).

**Theorem 3.** Let conditions of Theorem 1 hold. If the admissible coefficients \( k_1(x), k_2(x) \) satisfy the condition \( k_1(x) \geq k_2(x) \), \( \forall x \in [0, 1] \), then the output data \( \tilde{f}_i(t) \), \( i = 1, 2 \), have the following property:

\[ \tilde{f}_1(t) = - k(0) u_x(0, t; k_1) \leq \tilde{f}_2(t) = - k(0) u_x(0, t; k_2), \quad \forall t \in [0, T]. \]

**Proof.** Consider the solution \( \varphi(x, t) = \varphi(x, t; p) \) of problem (18) corresponding to input \( p(t) \). Assume that the function \( p(t) \) is positive on \( (0, T) \). Then, by using Theorem 1, we can show that \( \varphi_x(x, t) \) is positive on \( \Omega_T \) (the proof of this result is similar to the proof of Theorem 1 under a reversal of the time; in this case we need to take \( p(t) \) for \( g(t) \), and \( \varphi_x(x, t) \) for \( u_x(x, t) \)). Further, Theorem 1 implies also that \( (u_2)_x(x, t) \) is negative on \( \Omega_T \), \( \forall \tau \in [0, T] \). Thus, by the condition \( \Delta k(x) = k_1(x) - k_2(x) \geq 0 \) from (17) we have

\[ \int_0^\tau p(t) \Delta \tilde{f}(t) \, dt \leq 0, \quad \forall \tau \in [0, T]. \]

This implies \( \Delta \tilde{f}(t) = \tilde{f}_1(t) - \tilde{f}_2(t) \leq 0, \forall t \in [0, T] \).

It follows from this theorem also that the input–output mapping is well defined since \( \Delta k = 0 \) implies \( \Delta \tilde{f} = 0 \).

**Theorem 4.** If conditions of Theorem 1 hold, then input–output mapping \( \Phi[\cdot] : K \rightarrow f \) is Lipschitz continuous in the following sense:

\[ \| \tilde{f}_1 - \tilde{f}_2 \|_0 \leq L \| k_1 - k_2 \|_\infty, \]

where \( L = \|(u_2)_x\|_0 \| \varphi_x \|_0 \), and \( \| \cdot \|_0 \) and \( \| \cdot \|_\infty \) are the \( L_2 \)-norm and sup-norm, correspondingly; \( \varphi(x, t) = \varphi(x, t; p) \).
Proof. Choosing the arbitrary (control) function $p(t)$ as

$$ p(t) = \frac{\tilde{f}_1(t) - \tilde{f}_2(t)}{\|\tilde{f}_1 - \tilde{f}_2\|_0}, \quad \tau \in (0, T], $$

and substituting in (17) we get

$$ \|\tilde{f}_1 - \tilde{f}_2\|_0 \leq \|k_1 - k_2\|_\infty \left| \int_{\Omega_T} (u_2)_x \varphi_x \, dx \, dt \right|. $$

This, with the boundedness of the right-hand side integral, completes the proof. \qed

Thus we have shown the strict monotonicity and continuity of the mapping $\Phi[\cdot]: K \to f$, which implies the existence and uniqueness of the solution of the inverse problem (1)–(2).

4. Inverse source problem with single Neumann measured data

Consider now the following initial boundary value problems:

$$ \begin{cases} 
 u_t(x, t) = (k(x)u_x(x, t))_x + F(x, t), & (x, t) \in \Omega_T, \\
 u(x, 0) = 0, & 0 < x < 1, \\
 u(0, t) = 0, & k(1)u_x(1, t) = 0, \quad 0 < t < T, 
\end{cases} \quad (20) $$

where the source function $F(x, t)$ satisfies the condition:

(C3) $F(x, t) \in C(\Omega_T)$.

The inverse source problem here consists of determining the unknown source term $F = F(x, t)$ from the Neumann measured data $f(t)$ at the boundary $x = 0$, defined by (2).

Denote by $u := u(x, t; F)$ the solution of the parabolic direct problem (20) for a given $F(x, t) \in \mathcal{F}$ where $\mathcal{F} \subset C(\Omega_T)$ the set of admissible source terms $F = F(x, t)$. Then the function

$$ \tilde{f}(t; F) := -k(0)u_x(0, t; F), \quad t \in (0, T], $$

is defined to be the Neumann output data. Then the inverse source problem can be formulated in the following operator form

$$ \Psi[F](t) = f(t), \quad t \in (0, T]. \quad (22) $$

The mapping $\Psi[\cdot]: \mathcal{F} \to f$, $\Psi[F](\cdot) := -k(0)u_x(0, \cdot; F)$, is defined to be the input–output or source term-to-data mapping.

Hence the inverse source problem with the Neumann measured output data $f(t)$ can be reduced to the problem of inverting the input–output map $\Psi: \mathcal{F} \mapsto f$.

The following lemma shows the relationship between the input $F \in \mathcal{F}$ and the output $\tilde{f}(t; F) = -k(0)u_x(0, t; F)$ data.

**Lemma 3.** Assume that $u_1(x, t) = u(x, t; F_1)$ and $u_2(x, t) = u(x, t; F_2)$ are solutions of the direct problem (20) corresponding to the admissible source terms $F_1(x, t), F_2(x, t) \in \mathcal{F}$. Suppose that $\tilde{f}_j(t) = -k(0)u_x(0, t; F_j), \; j = 1, 2$, is the corresponding output, and let $\Delta \tilde{f}(t) =$
\[ \tilde{f}_1(t) - \tilde{f}_2(t), \quad \Delta F(x, t) = F_1(x, t) - F_2(x, t). \] Then for each \( \tau \in (0, T] \) the following integral identity holds:

\[ \int_0^\tau p(t) \Delta \tilde{f}(t) \, dt = \int_{\Omega_\tau} \int \Delta F(x, t) \varphi(x, t) \, dx \, dt, \]

where the function \( \varphi(x, t) = \varphi(x, t; p) \) is the solution of the following adjoint problem:

\[
\begin{aligned}
\varphi_t + (k(x)\varphi_x)_x &= 0, \quad (x, t) \in (0, 1) \times (0, \tau), \\
\varphi(x, \tau) &= 0, \quad x \in (0, 1), \\
\varphi(0, t) &= p(t), \quad \varphi_x(1, t) = 0, \quad t \in (0, \tau),
\end{aligned}
\]

with arbitrary (positive or negative) data \( p(t) \in C(0, T], \ p(\tau) = 0. \)

**Proof.** Let \( w(x, t) = u_1(x, t) - u_2(x, t) \). Then by Eq. (20), \( w_t = (u_1)_t - (u_2)_t = (u_1)_x - (u_2)_x = F_1(x, t) - F_2(x, t) \). Hence \( w = w(x, t) \) solves the following initial boundary value problem:

\[
\begin{aligned}
w_t - (k(x)w_x)_x &= \Delta F(x, t), \quad (x, t) \in (0, 1) \times (0, \tau), \ 0 < \tau \leq T, \\
w(x, 0) &= 0, \quad x \in (0, 1), \\
w(0, t) &= 0, \quad k(1)w_x(1, t) = 0, \quad t \in (0, \tau).
\end{aligned}
\]

Multiply each side of the above equation by an arbitrary function \( \varphi(x, t) \) and integrate by parts on \( \Omega_\tau = (0, 1) \times (0, \tau), \ 0 < \tau \leq T \):

\[
\int_{\Omega_\tau} \left[ w_t - (k(x)w_x)_x \right] \varphi \, dx \, dt + \int_{\Omega_\tau} w \left[ \varphi_t + (k(x)\varphi_x)_x \right] \, dx \, dt
\]

\[
= \int_0^\tau w \varphi_0^1 \, dx - \int_0^\tau (\varphi k w_x - w k \varphi_x)|_{x=1}^1 \, dt.
\]

Now we require that the function \( \varphi = \varphi(x, t) \) is the solution of the adjoint problem (24). Then due to the homogeneous boundary conditions \( w(x, 0) = w(0, t) = 0 \) the above integral identity implies:

\[
\int_{\Omega_\tau} \int \Delta F(x, t) \varphi(x, t) \, dx \, dt = \int_0^\tau \Delta \tilde{f}(t) p(t) \, dt.
\]

We have the proof. \( \Box \)

**Theorem 5.** If the admissible source terms \( F_1(x, t), F_2(x, t) \) satisfy the condition \( F_1(x, t) \geq F_2(x, t), \ \forall (x, t) \in \Omega_T, \) then the output data \( \tilde{f}_i(t), i = 1, 2, \) have the following property:

\[
\tilde{f}_1(t) = -k(0)u_x(0, t; F_1) \geq \tilde{f}_2(t) = -k(0)u_x(0, t; F_2), \quad \forall t \in [0, T].
\]

**Proof.** Consider the solution \( \varphi(x, t) = \varphi(x, t; p) \) of problem (24) corresponding to input \( p(t) \).

Without loss of generality we assume that the function \( p(t) \) is positive on \( (0, T) \). By using the maximum principle, we can show that \( \varphi(x, t) \) is positive on \( \Omega_T \). Hence we have
\[
\int_0^\tau \Delta \tilde{f}(t) p(t) \, dt \geq 0, \quad \forall \tau \in (0, T], \forall p(t) > 0.
\]

This implies \( \Delta \tilde{f}(t) = \tilde{f}_1(t) - \tilde{f}_2(t) \geq 0, \forall t \in [0, T] \). \( \Box \)

Since \( \Delta F = 0 \) implies \( \Delta f = 0 \), the input–output mapping \( \Psi[\cdot]: F \to f \) is well defined. Moreover, as shows the following result, this mapping is also Lipschitz continuous.

**Theorem 6.** If conditions of Theorem 1 hold, then input–output mapping \( \Psi[\cdot]: F \to f \) is Lipschitz continuous in the following sense:

\[
\| \tilde{f}_1 - \tilde{f}_2 \|_0 \leq L \| F_1 - F_2 \|_0,
\]

where \( L = \| \varphi \|_0 \).

**Proof.** Let us define the arbitrary (control) function \( p(t) \) in (23) as follows

\[
p(t) = \frac{\tilde{f}_1(t) - \tilde{f}_2(t)}{\| \tilde{f}_1 - \tilde{f}_2 \|_0}, \quad \tau \in (0, T].
\]

Then we get

\[
\| \tilde{f}_1 - \tilde{f}_2 \|_0 \leq \int \int \Omega_\tau \Delta F(x, t) \varphi \, dx \, dt \leq L \| F_1 - F_2 \|_0,
\]

which completes the proof. \( \Box \)

Thus we have shown the strict monotonicity and continuity of the mapping \( \Psi[\cdot]: F \to f \). This implies existence and uniqueness of the solution of the inverse source problem.

### 5. Numerical illustration

In this section we are going to illustrate just the usefulness of the integral identity (17) for numerical recovery of the unknown coefficient \( k(x) \). As a numerical algorithm we use coarse–fine mesh method, given in [10].

The piecewise-linear approximation

\[
k_I(x) = k_0(x) + \sum_{m=1}^{N_c} k_m \lambda_m(x), \quad 0 \leq x \leq 1,
\]

of the unknown diffusion coefficient \( k(x) \) is performed on the coarse space mesh \( \overline{W}_H := \{ x^c_m: x^c_0 = 0, \, x^c_m = mh_c, \, m = 1, N_c + 1, \, h_c = 1/(N_c + 1) \} \) (Fig. 1), by using the piecewise linear Lagrange basic functions (Fig. 2(b))

\[
\lambda_m(x) = \begin{cases} 
(x - x^c_{m-1})/(x^c_m - x^c_{m-1}), & x \in [x^c_{m-1}, x^c_m], \\
(x^c_{m+1} - x)/(x^c_{m+1} - x^c_m), & x \in [x^c_m, x^c_{m+1}], \\
0, & x \notin [x^c_{m-1}, x^c_{m+1}].
\end{cases}
\]

Here \( k_0(x) = k(0)(1 - x) + k(1)x, \, x \in [0, 1], \) is the given linear polynomial, according to Lemma 1.
The parameters $\kappa_m$ in (25) are defined as a difference between the values of the interpolant $k_I(x)$ and the linear function $k_0(x)$ at the coarse mesh points $x_m^c \in \mathbb{W}_c^c$: $\kappa_m = k_I(x_m^c) - k_0(x_m^c)$. Note that, the reason of the representation (25) via the difference $\kappa_m$ (not via the unknown parameters $k_m := k(x_m^c)$) is the presence of the term $\Delta k = k_1(x) - k(x_2)$ in integral identity (17).

The iteration process for the reconstruction of the piecewise-linear function (25) is organized starting from the first coarse mesh point $x_1^c$ (Fig. 2(a)), as follows:

$$k_m(x) = k_0(x) + \sum_{l=1}^{m-1} \kappa_l \lambda_l(x) + \kappa_m L_m(x), \quad m = 1, N_c, \quad 0 \leq x \leq 1,$$

where
\[ L_m(x) = \begin{cases} 
(x - x^c_{m-1})/(x^c_m - x^c_{m-1}), & x \in [x^c_{m-1}, x^c_m], \\
(1 - x)/(1 - x^c_m), & x \in [x^c_m, 1], \\
0, & x \in [0, x^c_{m-1}], 
\end{cases} \]

is the piecewise linear function corresponding to the coarse mesh point \( x^c_m \in W_{hc} \) (Fig. 2(a)–(b)).

At each \( m \)th step of the coarse mesh iteration process Lemma 2 with the integral identity (17) is used by taking the two iterations \( k_m(x), k_{m-1}(x) \) instead of \( k_1(x), k_2(x) \), and substituting \( \Delta k_m(x) = k_m(x) - k_{m-1}(x) \):

\[
\int \int_{\Omega} \Delta k_m(x)(u(x, t; k_{m-1}))_x (\varphi(x, t; k_m))_x \, dx \, dt = \int_{0}^{\tau_c^c} p_m(t) \Delta \tilde{f}(t) \, dt, \tag{27}
\]

where \( \Delta \tilde{f}(t) = \tilde{f}(t; k_m) - \tilde{f}(t; k_{m-1}) \). The final time \( \tau = \tau_c^c \) in this nonlinear integro-differential equation is taken to be the mesh point of the uniform coarse time mesh

\[ W_{T_c} := \{ \tau^c_m \in (0, T]: \tau^c_0 = 0, \tau^c_m = mT_c, \, m = 1, N_c + 1, \, T_c = T/(N_c + 1) \}, \]

which has the same number of points with the coarse space mesh \( W_{hc} \). The function \( \varphi(x, t; k_m) \) on the left-hand side of (27) is the solution of the adjoint problem

\[
\begin{align*}
\varphi_t + (k_m(x) \varphi_x)_x &= 0, & (x, t) \in (0, 1) \times (0, \tau_c^c), \\
\varphi(x, \tau) &= 0, & x \in (0, 1), \\
k_m(0) \varphi_x(0, t) &= p_m(t), & k_m(1) \varphi_x(1, t) = 0, & t \in (0, \tau_c^c), 
\end{align*} \tag{28}
\]

where

\[ p_m(t) = \begin{cases} c_p, & 0 \leq t \leq \tau_c^{m-1}, \\
c_p (\tau_c^c - t)/(\tau_c^c - \tau_c^{m-1}), & \tau_c^{m-1} \leq t \leq \tau_c^c. \end{cases} \tag{29}\]

Since the choice of the constant \( c_p \) in (29) does not have any effect on the algorithm, we take \( c_p = 1 \) for simplicity.

Let us transform Eq. (27). Taking into account (26) we may rewrite the term \( \Delta k_m(x) = k_m(x) - k_{m-1}(x) \) in the form \( \Delta k_m(x) = [\lambda_{m-1}(x) - L_{m-1}(x)]_x \kappa_{m-1} + L_m(x) \kappa_m, m = 2, 3, \ldots, N_c \).

Note that for \( m = 1 \), \( \Delta k_1(x) = k_1L_1(x) \). Substituting this in (27) we obtain the following nonlinear algebraic equation with respect to the unknown parameter \( \kappa_m \):

\[ M_m(\kappa_m) \kappa_m + \mathcal{N}_m(\kappa_m) = d_m, \quad m = 1, 2, \ldots, N_c. \]

Here the nonlinear functionals \( M_m = M_m(\kappa_m), \mathcal{N}_m = \mathcal{N}_m(\kappa_m) \) and the right-hand side \( d_m \) are defined by the following integrals:

\[
\begin{align*}
M_m(\kappa_m) &= \int_{0}^{\tau_c^c} \int_{0}^{1} L_m(x)(u(x, t; k_{m-1}))_x (\varphi(x, t; k_m))_x \, dx \, dt, \\
\mathcal{N}_m(\kappa_m) &= \kappa_{m-1} \int_{0}^{\tau_c^c} \int_{0}^{1} [\lambda_{m-1}(x) - L_{m-1}(x)](u(x, t; k_{m-1}))_x (\varphi(x, t; k_m))_x \, dx \, dt, \\
d_m &= \int_{0}^{\tau_c^c} p_m(t) \Delta f(t) \, dt.
\end{align*}
\]

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(a) Noise free data.

Fig. 3. Recovery of the coefficient $k(x) = 3 + 2x^4 - e^{x^2} + \arccos(x)$.

The nonlinear algebraic equation was solved by the simple iteration and bisection methods. In the both cases the stopping parameter $\delta_k := |\kappa^{(n)} - \kappa^{(n-1)}|$ was taken to be $\delta_k = 0.01$. The results obtained by the both algorithms were almost the same.

The synthetic measured data $f(t)$ was generated from the numerical solution of the parabolic problem (1) with the given coefficient $k(x) = \arccos x - \exp(-x^2) + 2x^4 + 3$ and the Dirichlet data $g(t) = t$. The obtained function $f(t) := k(0)u_x(0, t)$ was then assumed be the measured output data in the inverse problem (1)–(2). Figure 3(a) illustrates results of reconstruction of the coefficient $k(x)$ on the coarse meshes with the parameters $N_c = 8$ and $N_c = 13$. Relative errors, defined by $\varepsilon_k = \|(k - k_h)/k\|_\infty \times 100\%$ are $\varepsilon_k = 1.9 \times 10^{-2}$ and $\varepsilon_k = 8.2 \times 10^{-2}$, respectively. As shows the figure, in the case of noise free measured data $f(t)$ the reconstruction is high enough.

In the case of the noisy output data $f_{\gamma}(t) := f(t) \pm \gamma f(t)$, generated from the above synthetic data $f(t)$, the results of computational experiments are shown in Fig. 3(b), on the coarse meshes with $N_c = 9$. The noise factor was taken to be $\gamma = 0.1\%$. More deteriorations naturally occur near the end point $x^c_{N_c}$ of the coarse mesh. The reason of this phenomenon is that the computational errors accumulated from the previously determined parameters $\kappa_m, m = 1, 2, \ldots, N_c - 1$, are included in the next iteration. Subsequently errors are compounded, and due to ill-conditionedness of the inverse problem, this leads to the deterioration at the point $x^c_{N_c}$. 


Fig. 3. (continued)

References

