Soft Path Connectedness on Soft Topological Spaces

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Abstract

There are some theories as soft topological space and some related concepts such as soft interior, soft closed, soft subspace, soft separation axioms in \cite{8} and soft connectedness, soft locally connectedness in \cite{7}. In this paper, we define soft path connectedness on soft topological space and continue investigating the properties of soft path connectedness which is fundamental result for further research on soft topology.

Key words: Soft topological space, soft path connectedness.

1 Introduction

Since there was no mathematical concept to solve complicated problems in the economics, engineering, and environmental areas, a soft set theory was firstly introduced by Molodtsov \cite{6} to deal with the various kinds of uncertainties in these problems. Many researchers have contributed towards the soft set theory and its applications in various fields, increasingly \cite{1, 2, 5, 9, 10}. Recently the notion of soft topological spaces was studied by Shabir and Naz \cite{8}. They also introduced the concepts of soft open sets, soft closed sets, soft interior, soft closure and soft separation axioms. In \cite{7}, soft connectedness, soft locally connectedness was investigated.

In this paper, we introduce some new concepts in soft topological spaces such as soft path connectedness and examine some properties and relations of this concept.
2 Preliminaries

We now recall some definitions.

Definition 2.1. ([6]) Let \(X\) be an initial universe and \(E\) be a set of parameters. The power set of \(X\) is denoted by \(P(X)\) and \(A\) is a non-empty subset of \(E\). A pair \((F, A)\) is called a soft set over \(X\), where \(F\) is a mapping given by \(F : A \rightarrow P(X)\). In other words, a soft set over \(X\) is a parameterized family of subsets of the universe \(X\). For \(e \in A\), \(F(e)\) can be considered as the set of \(e\)-approximate elements of the soft set \((F, A)\). Clearly, a soft set is not a set.

Definition 2.2. ([6]) The intersection of two soft sets \((F, A)\) and \((G, B)\) over \(X\), is the soft set \((H, C)\), where \(C = A \cap B\), and \(\forall c \in C\), \(H(e) = F(e) \cap G(e)\). We write \((F, A) \cap (G, B) = (H, C)\).

Definition 2.3. ([6]) A soft set \((F, A)\) over \(X\) is called a null soft set, denoted by \(\Phi\), if \(\forall e \in A\), \(F(e) = \emptyset\).

Definition 2.4. ([6]) A soft set \((F, A)\) over \(X\) is called an absolute soft set, denoted by \(A\), if \(\forall e \in A\), \(F(e) = X\).

Definition 2.5. ([6]) The union of two soft sets \((F, A)\) and \((G, B)\) over \(X\), is the soft set \((H, C)\), where \(C = A \cup B\), and \(\forall c \in C\)

\[
H(e) = \begin{cases} 
F(e), & \text{if } e \in A - B, \\
G(e), & \text{if } e \in B - A, \\
F(e) \cup G(e), & \text{if } e \in A \cap B.
\end{cases}
\]

We write \((F, A) \cup (G, B) = (H, C)\).

Definition 2.6. ([5]) Let \((F, A)\) and \((G, B)\) be two soft sets over \(X\). Then \((F, A)\) is a subset of \((G, B)\), denoted by \((F, A) \subseteq (G, B)\) if \(A \subseteq B\) and for all \(e \in E\), \(F(e) \subseteq G(e)\).

Definition 2.7. ([3]) A soft set \((F, E)\) is called a soft point, denoted by \((x_e, E)\), if for each element \(e \in E\), \(F(e) = \{x\}\) and \(F(e') = \emptyset\) for all element \(e' \in E - \{e\}\).

Definition 2.8. ([8]) Let \(\tau\) be the collection of soft sets over \(X\), then \(\tau\) is called a soft topology on \(X\) if \(\tau\) satisfies the following axioms:

(i) \(\Phi, X\) belong to \(\tau\).
(ii) The union of any number of soft sets in \(\tau\) belongs to \(\tau\).
(iii) The intersection of any number of soft sets in \(\tau\) belongs to \(\tau\).

The triplet \((X, \tau, E)\) is called a soft topological space over \(X\). The members of \(\tau\) are said to be soft open in \(X\).
Definition 2.9. ([4]) A soft set \((G, E)\) in a soft topological space \((X, \tau, E)\) is called a soft neighborhood of \(x \in X\) if there exists a soft open set \((F, E)\) such that \(x \in (F, E) \subseteq (G, E)\).

Definition 2.10. ([10]) Let \((X, \tau, E)\) and \((Y, \tau', E)\) be two soft topological spaces, \(f : (X, \tau, E) \to (Y, \tau', E)\) be a mapping. For each neighborhood \((H, E)\) of \((f(x), E)\), if there exists a soft neighborhood \((F, E)\) of \((x, E)\) such that \(f((F, E)) \subseteq (H, E)\), the \(f\) is called a soft continuous mapping at \((x, E)\).

Definition 2.11. ([7]) Let \((X, \tau, E)\) be a soft topological space over \(X\). A soft separation of \(X\) is a pair \((F, E)\) and \((G, E)\) of non-null soft open sets over \(X\) such that
\[
\bar{\mathcal{X}} = (F, E) \cup (G, E); \quad (F, E) \cap (G, E) = \emptyset
\]

Definition 2.12. ([7]) A soft topological space \((X, \tau, E)\) is said to be soft connected if there does not exist a soft separation of \(X\).

Definition 2.13. ([7]) Let \(\{(X_s, \tau_s, E_s)\}_{s \in S}\) be a family of soft topological spaces and define
\[
B = \left\{ \prod_{s \in S} (F_s, E_s) : (F_s, E_s) \in \tau_s \right\}
\]
and \(\tau\) as the collection of all arbitrary union of elements of \(B\) and \(\tau\) is a soft topology over \(\prod_{s \in S} (X_s, \tau_s, E_s)\).

Unless otherwise stated \(E = \mathbb{N} \cup \{0\}\) will be assumed to be a set of parameters and the set of rational numbers on the closed interval \(I = [0, 1]\) will be considered as \(\{0, 1, r_3, r_4, r_5, \ldots\}\). If \(e \in E, r_e \in \mathbb{Q} \cap I\) and for all \(\varepsilon > 0\) we define the soft set \(F_\varepsilon : E \to \mathcal{P}(I)\) as \(F_\varepsilon(e) = (r_e - \varepsilon, r_e + \varepsilon)\). Then the family \(B = \{(F_\varepsilon, E)\}_{\varepsilon > 0}\) is a soft base of soft topology on \(I\). Then \(\tau_I\) is called a soft topology generated by \(B\).

Definition 2.14. A soft topological space \((I, \tau_I, E)\) is called a unit soft interval.

Definition 2.15. Let \((I, \tau_I, E)\) be a unit soft interval and \((X, \tau, E')\) be a soft topological space. A soft path is a soft continuous map \(f, \varphi : (I, \tau_I, E) \to (X, \tau, E')\) The soft sets \(\{f(0), \varphi(e)\}_{e \in E}\) and \(\{f(1), \varphi(e)\}_{e \in E}\) are said to be the initial and final of the soft path \((f, \varphi)\) where \(f : I \to X, \varphi : E \to E'\). It’s clear that for each \(e \in E\), the map \(f : (I, \tau_I) \to (X, \tau, E')\) is a path from \(f(0), \varphi(e)\) to \(f(1), \varphi(e)\). Hence every soft path can be considered as a parametrized family on the soft topological space \((X, \tau, E')\).

Definition 2.16. Let \((I, \tau_I, E)\) be a unit soft interval and \((X, \tau, E')\) be a soft topological space. \((X, \tau, E')\) is said to be a soft path connected space if for each soft points \((x_{e_1}', E'), (y_{e_2}', E')\), there exists a soft path \((f, \varphi : (I, \tau_I, E) \to (X, \tau, E')\) such that \(\varphi(e_1) = e_1', \varphi(e_2) = e_2', f(0) = x, f(1) = y\).
3 The Main Results

The following proposition shows the relation between the soft path connected topological space and path connected topological space.

**Proposition 3.1.** If \((X, \tau, E')\) is a soft path connected topological space, then \((X, \tau_{e'})\) is a path connected topological space for each \(e' \in E'\).

**Proof.** Let \(x, y \in (X, \tau_{e'})\) be arbitrary points; then \((x_{e'}, E'), (y_{e'}, E') \in (X, \tau, E')\) are soft points. Since \((X, \tau, E')\) is a soft path connected topological space, there exists a soft path \((f, \varphi) : (I, \tau_I, E) \to (X, \tau, E')\) such that

\[
\varphi(e) = e', f(0) = x, f(1) = y.
\]

So for each \(e' \in E'\), we now have a path from \(x\) to \(y\) defined by

\[
f_e : (I, (\tau_I)_e) \to (X, \tau_{e'})
\]

This implies that \((X, \tau_{e'})\), \(\forall e' \in E'\) is a path connected topological space. \(\square\)

**Proposition 3.2.** The soft unit interval \((I, \tau_I, E)\) is a soft path connected topological space.

**Proof.** Since \((1_{I}, 1_E) : (I, \tau_I, E) \to (I, \tau_I, E)\) is a soft continuous map, this is trivial. \(\square\)

The converse of Proposition 3.1 is is not always true, as shown in the next example.

**Example 3.3.** Let \((X, \tau_1)\) and \((Y, \tau_2)\) be two path connected and disjoint topological spaces. Then for \(E_1 = \{e_1\}, E_2 = \{e_2\}\) we can construct the following soft topological spaces

\[
\{F_U : E_1 \to P(X) : F_U (e_1) = U, U \in \tau_1\}
\]

\[
\{F_V : E_2 \to P(Y) : F_V (e_2) = V, V \in \tau_2\}
\]

and consider the soft topological space \((X \oplus Y, \tau_1 \oplus \tau_2, E_1 \cup E_2)\). For any \(c_1 \in E_1 \cup E_2\), \((X \oplus Y, (\tau_1 \oplus \tau_2)_{c_1}) = (X, \tau_1)\) and \((X \oplus Y, (\tau_1 \oplus \tau_2)_{c_2}) = (Y, \tau_2)\) are path connected topological spaces but \((X \oplus Y, \tau_1 \oplus \tau_2, E_1 \cup E_2)\) is not a soft path connected topological space.

**Theorem 3.4.** The image under a soft continuous map of a soft path connected topological space is soft path connected.

**Proof.** Suppose \((X, \tau, E')\) is a soft path connected topological space, \((Y, \tau', E'')\) is a soft topological space and \((g, \psi) : (X, \tau, E') \to (Y, \tau', E'')\) is a soft continuous map. Let \((y_{e''}, E'')\) and \((\overline{y}_{e'}, E''\)) be soft points of the image \((g, \psi) (X, \tau, E')\); then there exist soft points \((x_{e'}, E'), (\overline{x}_{e'}, E') \in (X, \tau, E')\) so that

\[
\psi (e') = e''_1, \psi (\overline{e'}) = \overline{e''}, g (x) = y, g (\overline{x}) = \overline{y}.
\]
Since \((X, \tau, E')\) is a soft path connected topological space, we have a soft path \((f, \varphi) : (I, \tau_I, E) \rightarrow (X, \tau, E')\) such that
\[
\varphi(e_1) = e'_1, \varphi(e_1) = e'_2, f(0) = x, f(1) = \overline{x}.
\]
Hence \((g, \psi) \circ (f, \varphi) : (I, \tau_I, E) \rightarrow (Y, \tau', E'')\) is a soft path from soft point \(\left(y_{e'_1}, E''\right)\) to soft point \(\left(y_{e'_2}, E''\right)\).

**Theorem 3.5.** The soft unit interval \((I, \tau_I, E)\) is a soft connected topological space.

**Proof.** We argue by contradiction. Suppose that the soft unit interval \((I, \tau_I, E)\) is not a soft connected topological space. Then
\[
(F, E) \cup (G, E) = (I, \tau_I, E)
\]
where \((F, E)\) and \((G, E)\) are nonempty, disjoint soft sets. For each \(e \in E\), choose nonempty sets \(F(e), G(e) \in (\tau_I)_e\) and then
\[
F(e) \cup G(e) = I, F(e) \cap G(e) = \emptyset.
\]
It follows that \((I, (\tau_I)_e)\) is not a connected topological space, which is a contradiction.

**Theorem 3.6.** Any soft path connected topological space is a soft connected topological space.

**Proof.** Assume for contradiction that \((X, \tau, E')\) is soft path connected topological space but not a soft connected topological space. Then there exist nonempty, disjoint soft sets \((F, E')\) and \((G, E')\) so that
\[
(F, E') \cup (G, E') = \overline{X}.
\]
Since \((X, \tau, E')\) is soft path connected topological space, for each soft points \(\left(x_{e'_1}, E'\right) \in (F, E'), \left(y_{e'_2}, E'\right) \in (G, E')\) there exists a soft path \((f, \varphi) : (I, \tau_I, E) \rightarrow (X, \tau, E')\) such that
\[
\varphi(e_1) = e'_1, \varphi(e_2) = e'_2, f(0) = x, f(1) = y.
\]
By Theorem 3.5, \((I, \tau_I, E)\) is a soft connected topological space and Theorem 2.8 in [7] implies that \((f, \varphi) (I, \tau_I, E)\) is a soft connected topological space. Suppose that
\[
(F_1, E') = (F, E') \cap (f, \varphi)(I, \tau_I, E) \quad (G_1, E') = (G, E') \cap (f, \varphi)(I, \tau_I, E)
\]
are two disjoint soft sets on the soft space \((f, \varphi) (I, \tau_I, E)\) such that
\[
(f, \varphi)(I, \tau_I, E) = (F_1, E') \cup (G_1, E')
\]
This contradicts the fact that \((f, \varphi)(I, \tau_I, E)\) is a soft connected topological space.
Theorem 3.7. The topological product of soft topological spaces is a soft path connected topological space if and only if each soft topological space is a soft path connected topological space.

Proof. Let \( \{(X_s, \tau_s, E_s)\}_{s \in S} \) be a family of soft topological spaces and \( \prod_{s \in S} (X_s, \tau_s, E_s) \) be a soft path connected topological space. Since the projections \((p_s, q_s) : \prod_{s \in S} (X_s, \tau_s, E_s) \to (X_s, \tau_s, E_s)\) are soft continuous and surjective, for each \( s \in S \), \( \{(X_s, \tau_s, E_s)\}_{s \in S} \) is a soft path connected topological space.

Conversely, Let \( \{(X_s, \tau_s, E_s)\}_{s \in S} \) be a family of soft path connected topological spaces and \( \{(x_s)_{s \in S}, E_s\} \) and \( \{(y_s)_{s \in S}, E_s\} \) be soft points of \( \prod_{s \in S} (X_s, \tau_s, E_s) \). Then for each \( s \in S \), \( (x_s)_{s \in S}, E_s \) and \( (y_s)_{s \in S}, E_s \) belong to \( (X_s, \tau_s, E_s) \). Since \( (X_s, \tau_s, E_s) \) is a soft path connected topological space, we have a soft path \( (f_s, \varphi_s) : (I, \tau_I, E) \to (X_s, \tau_s, E_s) \) such that

\[
\varphi_s(e) = e_s, \varphi_s(\bar{e}) = \bar{e}_s, f_s(0) = x_s, f_s(1) = y_s.
\]

If we now take \( \varphi : E \to \prod_{s \in S} E_s, f : I \to \prod_{s \in S} X_s \) defined by \( \varphi(e) = \{\varphi_s(e)\}_{s \in S}, f(x) = \{f_s(x)\}_{s \in S} \), then \( (f, \varphi) : (I, \tau_I, E) \to \prod_{s \in S} (X_s, \tau_s, E_s) \) is a soft path so that

\[
\varphi(e) = \{\varphi_s(e)\}_{s \in S} = \{e_s\}_{s \in S}, \varphi(\bar{e}) = \{\varphi_s(\bar{e})\}_{s \in S} = \{\bar{e}_s\}_{s \in S}, \\
f(0) = \{f_s(0)\}_{s \in S} = \{x_s\}_{s \in S}, f(1) = \{f_s(1)\}_{s \in S} = \{y_s\}_{s \in S}.
\]

References


