Characteristic Equation-Based Computation of Thévenin and Norton Equivalent Circuits

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In this paper, a new method is proposed for determining the parameters of Thévenin and Norton equivalent circuits. The method is unified and employs only one topology that depends on the test load impedance, \( Z_r \), connected to the output of the circuit. It does not require setting all independent sources to zero for determining Thévenin and Norton impedance. All equivalent circuit parameters are derived simultaneously and systematically from the characteristic equation relating to the output of the circuit. © 2011 Institute of Electrical Engineers of Japan. Published by John Wiley & Sons, Inc.

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1. Introduction

The Thévenin and Norton theorems are of high importance in electrical and electronics circuit analysis. Any linear circuit, no matter how complex, may be represented by Thévenin or Norton equivalent circuits (1–4). The parameters of these equivalent circuits (i.e., \( U_{th} \), \( Z_{th} \), \( J_{th} \), and \( Z_{th} \)) are determined by various approaches in circuit analysis. The Thévenin voltage, \( U_{th} \), is the open circuit voltage at the desired terminal pair of a given circuit. The Norton current, \( J_{nor} \), is the short circuit current at the desired terminal pair of a given circuit. The Thévenin/Norton impedance, \( Z_{th} = Z_{nor} \), whose computation is extremely time-consuming, is the equivalent impedance at the desired terminal pair by setting all independent sources to zero. If there are dependent sources in the circuit, test current or test voltage source is applied to the circuit where all independent sources are set to zero. Hereby, the Thévenin/Norton impedance is found by using the voltage and current of these test sources.

There are some studies about Thévenin and Norton theorems in the literature. The systematic structure of Thévenin theorem for linear \( n \) port networks was given in Ref. (5). Some computational methods for Thévenin and Norton equivalents of the circuits were investigated in Refs. (6–9). Mood presented the concepts of the Thévenin and Norton theorems and surveyed some textbooks presenting these theorems and gave methods of finding equivalent circuits for two ports and multiterminal networks (10). Haley gave proof of the Thévenin theorem for linear circuits using current–voltage (\( I \sim V \)) characteristic of the output of the circuits (11). This method needs no network equations or special circuits, but it is convenient only for experimental applications. Jin and Chan presented a unified method in which two parameters of Thévenin and Norton equivalent circuits can be obtained simultaneously and systematically without requiring setting all dependent sources to zero (12). But, their method requires using the current source as a test source to obtain the Thévenin equivalent circuit and the voltage source as a test source to obtain the Norton equivalent circuit. In addition to this restriction, the experimental application of method is limited because there is no physical current source for experimental applications. Bogard introduced an experimental approach to find the output resistance, similar to the equivalent resistance, of the amplifier circuits (13). The applications of Thévenin and Norton’s theorems when dealing with special cases of electric circuits are demonstrated in Ref. (14). A novel analytical approximation method of frequency dependent Thévenin impedance is given in Ref. (15). A laboratory exercise is presented that facilitates the teaching of Thévenin’s equivalent circuit and maximum power transfer through the use of a black box equipped with two external terminals in Ref. (16). A novel Thévenin equivalent calculation based on varying system condition is analyzed in Ref. (17). Recursive least square estimation technique is applied to estimate online Thévenin equivalent and track system voltage stability. Different engineering applications relating to Thévenin equivalent circuits are given in Refs. (18–21).

In circuit analysis, different topologies are used to derive the parameters of Thévenin and Norton equivalent circuits. In order to find \( U_{th} \), the open circuit voltage at the desired terminals is calculated. Similarly, to find \( J_{nor} \), the desired terminals are short circuited and the current through this short circuit is calculated. In determining of Thévenin and Norton impedances, \( Z_{th} = Z_{nor} \), the equivalent impedance at the desired terminals is calculated by setting all independent sources to zero. The use of different topologies in obtaining the parameters of equivalent circuits is extremely time-consuming.

This paper proposes a novel unified, efficient, and easy understandable approach for computation of equivalent circuit parameters. The method employs only one topology that depends on the test load impedance, \( Z_r \), connected to the output of the circuit. It does not require setting all independent sources to zero for determining Thévenin and Norton impedance. The parameters of the Thévenin and Norton equivalent circuits are derived simultaneously and systematically from the characteristic equation relating to the output terminals of the circuit, which mainly depends on test load impedance. The proposed method is more powerful and systematic than the conventional methods because it uses one topology to obtain all equivalent circuit parameters. Another important power of the method is convenient for realizing experimentally.

The following two sections explain the basic form and the generalized state of the proposed method, respectively. Section 4
illustrates its four application examples, followed by final remarks in the last section.

2. Fundamentals of the Proposed Method

Let \( N \) be a linear time-invariant circuit, which may include dependent and independent sources, magnetic coupling. \( N \) may be either resistive or dynamic. Suppose a port is created in \( N \), as shown in Fig. 1, and let us obtain the Thévenin or Norton equivalent circuits for this port.

The port terminals are denoted as nodes a and b. Thévenin and Norton equivalent circuits of the basic circuit are given in Fig. 2(a) and (b), respectively.

The parameters of Thévenin and Norton equivalent circuits relating to any circuit can be found by determining the open-circuit voltage \( U_{oc} \) and the short-circuit current \( I_{sc} \) obtained for a desired port of basic circuit (a and b in Fig. 1).

Thévenin voltage:

\[
U_{th} = U_{oc} \quad (1)
\]

Norton current:

\[
I_{n} = I_{sc} \quad (2)
\]

Thévenin/Norton impedance:

\[
Z_{th} = Z_{n} = \frac{U_{oc}}{I_{sc}} = \frac{U_{th}}{I_{n}} \quad (3)
\]

In this paper, we introduce a new approach employing test load impedance, \( Z_{x} \), to determine the parameters of Thévenin and Norton equivalent circuits. This section details how these parameters can be expressed in terms of \( Z_{x} \) and Norton equivalent circuits. The port voltage \( U_{ab} \) and the port current \( I_{ab} \) according to Fig. 2(a) and (b) are obtained as follows:

The port voltage:

\[
U_{ab} = \frac{Z_{x}}{Z_{th} + Z_{x}} U_{th} \quad (4a)
\]

or

\[
U_{ab} = \frac{Z_{n} Z_{x}}{Z_{th} + Z_{n}} I_{n} \quad (4b)
\]

The port current:

\[
I_{ab} = \frac{U_{th}}{Z_{th} + Z_{x}} \quad (5a)
\]

or

\[
I_{ab} = \frac{Z_{n} I_{n}}{Z_{th} + Z_{n}} \quad (5b)
\]

Norton current: In (5a) and (5b), the limit of the \( I_{ab} \) as \( Z_{x} \) approaches zero is equal to the short-circuit current \( (U_{ab} = 0) \). In this case, the Norton current is expressed as below:

\[
I_{sc} = I_{n} = \lim_{Z_{x} \to 0} I_{ab} \quad (6)
\]

Thévenin voltage: In (4a) and (4b), the limit of the \( U_{ab} \) as \( Z_{x} \) approaches infinite is equal to the open-circuit voltage \( (I_{ab} = 0) \). In this case, the Thévenin voltage is expressed as below:

\[
U_{th} = U_{oc} = \lim_{Z_{x} \to \infty} U_{ab} \quad (7)
\]

Thévenin/Norton impedance: Suppose that the parameters of both circuits are fixed and the test load impedance \( Z_{x} \) is adjustable in Fig. 2(a) and (b). For determining the Thévenin (or Norton) impedance, we extract the expression of the test load impedance \( Z_{x} \) from (4a) as below:

\[
Z_{x} = \frac{U_{ab}}{U_{th} - U_{ab}} Z_{th} \quad (8)
\]

In Fig. 2(a), when the port voltage \( U_{ab} \) drops from open-circuit voltage to one-half of it \( (U_{ab} = U_{th}/2) \), the test load impedance is equal to the Thévenin impedance by the voltage divider rule. If this condition is substituted into (8), it is seen that the Thévenin impedance is equal to the test load impedance as in (9).

This approach is used to determine experimentally the output resistance of the amplifier circuits in electronic experiments (10):

\[
Z_{x} = \frac{U_{th}/2}{U_{th} - U_{th}/2} Z_{th} \to Z_{x} = Z_{th} \quad (9)
\]

Similarly, this impedance can also be expressed according to the Norton equivalent circuit. First, the expression of the test load impedance, \( Z_{x} \), is extracted from (5b) as below:

\[
Z_{x} = \frac{Z_{n}}{I_{n}} \quad (10)
\]

In Fig. 2(b), when the port current \( I_{ab} \) drops from short-circuit current to one-half of it \( (I_{ab} = I_{sc}/2) \), the test load impedance is equal to the Norton impedance by the current divider rule. If this condition is substituted into (10), the Norton impedance is obtained as:

\[
Z_{x} = \frac{Z_{n}}{I_{sc}/2} \quad (11)
\]

Consequently, each one of the below equations gives the Thévenin (or Norton) impedance:

\[
Z_{th} = \lim_{U_{ab} \to U_{th}/2} Z_{x} \quad \text{or} \quad Z_{n} = \lim_{I_{ab} \to I_{sc}/2} Z_{x} \quad (12)
\]

3. Generalized State of the Proposed Method

We now explain the generalized state of the proposed approach for given equivalent circuits in Fig. 2(a) and (b). The method used for determination of the Thévenin or Norton equivalent circuits of general circuit in Fig. 3 is stated as below.

The test load impedance \( Z_{x} \) is connected into the port. The port voltage \( U_{ab} \) and the port current \( I_{ab} \), including the test load impedance, are generated by using any formulation method such
The system equations: \( AX(s) = BU(s) \) \hspace{1cm} (13)

The output equations: \( Y(s) = CX(s) + DU(s) \) \hspace{1cm} (14)

where \( A, B, C, D \) are coefficient matrices, \( U(s) \) is the source vector, \( X(s) \) is the unknown vector, \( Y(s) \) is the output vector. Matrix \( A \) is also called the characteristic matrix in the circuit analysis. Solutions of the system equations and the output equations are given in (15) and (16), respectively:

\[
X(s) = A^{-1}BU(s) = \left( \frac{1}{\det(A)} \text{Adj}(A) \right) BU(s) \tag{15}
\]

\[
Y(s) = CA^{-1}BU(s) + DU(s) = \left[ CA^{-1}B + D \right] U(s) = \left[ \frac{1}{\det(A)} \text{Adj}(A) \right] B + D \tag{16}
\]

It is obvious that solutions of (15) and (16) are fractional. The determinant of the characteristic matrix, \( A \), has also fractional and polynomial form as in (17):

\[
\det(A) = \frac{Q(s)}{R(s)} \tag{17}
\]

where \( Q(s) \) and \( R(s) \) show the numerator and the denominator of the determinant, respectively. The determinant expression in (17) is substituted into (15) and (16):

\[
X(s) = \left( \frac{1}{\frac{Q(s)}{R(s)}} \text{Adj}(A) \right) BU(s) = \frac{\text{Adj}(A) * B * R(s)}{Q(s)} U(s) = \frac{M(s)}{Q(s)} U(s) \tag{18}
\]

where

\[
M(s) = \text{Adj}(A) * B * R(s)
\]

\[
Y(s) = \left[ C * \frac{1}{\frac{Q(s)}{R(s)}} \text{Adj}(A) \right] B + D \frac{U(s)}{Q(s)} = \frac{N(s)}{Q(s)} U(s) \tag{19}
\]

where

\[
N(s) = C * \text{Adj}(A) * B * R(s) + D * Q(s)
\]

The transfer functions of the system are obtained from (19):

\[
H(s) = \frac{Y(s)}{U(s)} = \frac{N(s)}{Q(s)} = C * M(s) + D * Q(s) \tag{20}
\]

The numerator of determinant of the coefficient matrix \( A \) in (17) and the denominator of the transfer functions in (20) are equal. Therefore, \( Q(s) \) polynomial is also called the characteristic equation in circuit analysis. The port voltage or the port current relating to the general circuit is expressed in terms of variables of the utilized method.

The equivalent circuit parameters relating to the general circuit have fractional and polynomial form as shown in (21):

\[
U_{ab}(s) = \frac{P_1(s)}{Q_1(s)} \tag{21a}
\]

\[
Z_{ab}(s) = Z_{loc}(s) = P_2(s) \tag{21b}
\]

\[
J_{loc}(s) = \frac{P_3(s)}{Q_3(s)} \tag{21c}
\]

For the general circuit, the port voltage in (4a) and the port current in (5a) can be presented as below in terms of the expressions in (21).

The port current:

\[
I_{ab}(s) = \frac{U_{th}(s)}{Z_{ab}(s)} + Z_t = \frac{P_1(s)}{Q_1(s)} \frac{P_1(s)}{Q_1(s)} + Z_t \tag{22}
\]

The port voltage:

\[
U_{ab}(s) = \frac{Z_t}{Z_{th}(s) + Z_t} U_{th}(s) = \frac{Z_t}{Z_{th}(s) + Z_t} P_1(s) = \frac{P(s) Z_t}{Q(s)} \tag{23}
\]

where

\[
P(s) = P_1(s)Q_2(s), \Delta(s) = P_2(s)Q_1(s), \Delta_t(s) = Q_1(s)Q_2(s)
\]

According to system theory, all circuit variables and transfer functions relating to any circuit have the same denominator in \( s \) domain (1–4). Therefore, the denominators of the transfer function in (20) and the port variables in (22) and (23) are equal to the characteristic equation \( Q(s) \) of the system. \( P(s) \) in the expressions of the output variables (the port voltage or the port current) is equal to \( N(s)/U(s) \) according to (19). \( \Delta(s) \) is the coefficients of \( Z_t \) and \( \Delta_t(s) \) consists of terms independent of \( Z_t \).

The denominators \( Q(s) \), characteristic equations, of (22) and (23) are identical, but their numerators are different. Moreover, the components \( (\Delta, \Delta_t) \) of characteristic equations are identical. The characteristic equation, \( Q(s) \), can be always partitioned in terms of \( \Delta, \Delta_t, Z_t \), as in (22) and (23). In order to determine the parameters of Thévenin and Norton equivalent circuits, our proposed approach depends on obtaining the components \( (\Delta, \Delta_t) \) of the characteristic equation and \( P(s) \) polynomial. Here, the most important point is that these components give directly the parameters of the equivalent circuits, which are indicated clearly below.

After expressing the port voltage \( U_{ab} \) or the port current \( I_{ab} \) in the general circuit, the parameters of equivalent circuits are determined as follows. Here, they are obtained by using concepts of the open-circuit voltage \( U_{oc} \) and short-circuit current \( I_{oc} \), as explained in Section 2. In short-circuit state, \( Z_t \) is zero. In open-circuit state, \( Z_t \) is infinite.

Norton current: In (22), the limit of \( I_{ab} \) as \( Z_t \) approaches zero is equal to the short-circuit current \( I_{oc} \) and this current is the Norton current:

\[
J_{loc}(s) = I_{oc}(s) = \lim_{Z_t \to 0} I_{ab}(s) = \frac{P(s)}{\Delta(s)} \tag{24}
\]

According to (23), \( \lim_{Z_t \to 0} U_{ab}(s) = 0. \)
Thévenin voltage: In (23), the limit of the $U_{ab}$ as $Z_s$ approaches infinite is equal to the open-circuit voltage ($U_{oc}$) and this voltage is the Thévenin voltage.

\[
U_{th}(s) = U_{oc}(s) = \lim_{Z_s \to \infty} U_{ab}(s) = \frac{\infty}{\infty} \quad (25)
\]

For definiteness in (25), if L’Hospital rule is used, the expression is obtained as:

\[
U_{th}(s) = U_{oc}(s) = \lim_{Z_s \to \infty} U_{ab}(s) = \frac{P(s)}{\Delta_s(s)} \quad (26)
\]

According to (22), \( \lim_{Z_s \to \infty} I_{ab}(s) = 0 \).

Thévenin/Norton impedance: The expression of the test load impedance is extracted from (22) as below:

\[
Z_s(s) = \frac{P(s)}{I_{ab}(s) \Delta_s(s)} - \frac{\Delta(s)}{\Delta_s(s)} \quad (27)
\]

When the port current of the general circuit is equal to one-half of its short-circuit current, the Thévenin/Norton impedance and test load impedance are equal. Hence, the limit of the test load impedance as $I_{ab}$ approaches one-half of $I_{oc}$ describes the Thévenin/Norton impedance:

\[
Z_{th}(s) = Z_{nec}(s) = \lim_{I_{ab} \to I_{oc}/2} Z_s(s) = \frac{P(s)}{\frac{P(s)}{\Delta_s(s)} \Delta_s(s)} - \frac{\Delta(s)}{\Delta_s(s)} \quad (28)
\]

If the expression of $I_{ab} in (24)$ is substituted into (28), $Z_{th}(Z_{nec})$ is obtained as below:

\[
Z_{th}(s) = Z_{nec}(s) = \frac{P(s)}{\frac{P(s)}{\Delta_s(s)} \Delta_s(s)} - \frac{\Delta(s)}{\Delta_s(s)} = \frac{\Delta(s)}{\Delta_s(s)} \quad (29)
\]

Equation (23) can be also used to express Thévenin/Norton impedance. The expression of the test load impedance is extracted from (23) as below:

\[
Z_s(s) = \frac{U_{ab}(s) \Delta_s(s)}{P(s) - U_{ab}(s) \Delta_s(s)} \quad (30)
\]

When the port voltage of the general circuit is equal to one-half of its open-circuit voltage, the Thévenin/Norton impedance and test load impedance are equal. Hence, the limit of the test load impedance as $U_{ab}$ approaches one-half of $U_{oc}$ describes the Thévenin/Norton impedance:

\[
Z_{th}(s) = Z_{nec}(s) = \lim_{U_{ab} \to U_{oc}/2} Z_s(s) = \frac{\frac{U_{ab}(s)}{\Delta_s(s)} \Delta_s(s)}{P(s) - \frac{U_{ab}(s)}{\Delta_s(s)} \Delta_s(s)} \quad (31)
\]

If the expression of $U_{oc}$ in (26) is substituted into (31), $Z_{th}(Z_{nec})$ is obtained as below:

\[
Z_{th}(s) = Z_{nec}(s) = \frac{P(s)}{\frac{P(s)}{\Delta_s(s)} \Delta_s(s)} - \frac{\Delta(s)}{\Delta_s(s)} = \frac{\Delta(s)}{\Delta_s(s)} \quad (32)
\]

The Thévenin/Norton impedance can be also obtained directly from (24) and (26):

\[
Z_{th}(s) = Z_{nec}(s) = \frac{U_{th}(s)}{I_{th}(s)} = \frac{\Delta(s)}{\Delta_s(s)} \quad (33)
\]

Moreover, it can be also realized that (24), (26), and (33) are equal to the expressions in (21), respectively:

\[
U_{th}(s) = \frac{P(s)}{\Delta_s(s)} = \frac{P_1(s)Q_2(s)}{Q_1(s)Q_2(s)} = \frac{P_1(s)}{Q_1(s)} \quad (34a)
\]

\[
J_{nec}(s) = \frac{P(s)}{\Delta(s)} = \frac{P_1(s)Q_2(s)}{Q_1(s)Q_2(s)} = \frac{P_1(s)}{Q_1(s)} \quad (34b)
\]

\[
Z_{th}(s) = Z_{nec}(s) = \frac{\Delta(s)}{\Delta_s(s)} = \frac{P_2(s)Q_1(s)}{Q_1(s)Q_2(s)} = \frac{P_2(s)}{Q_2(s)} \quad (34c)
\]

After connecting the test load impedance ($Z_s$) into the general circuit and expressing the port voltage or the port current according to this element, the characteristic equation and the required components ($\Delta, \Delta_s, P$) are easily determined. All equivalent circuit parameters are simultaneously derived in terms of $\Delta, \Delta_s, P$ from only one topology. Therefore, the proposed method is unified and systematic, which has been applied to all kind of possible complicated circuit examples.

4. Applications

In this section, four application examples are presented. First example, a simple resistive circuit in Fig. 4, is taken to show systematical and experimental power of the method. Besides, the method is compared with the conventional methods from the point of theoretical and experimental.

Example 1. The proposed method requires to be connected a test load resistance, potentiometer, to designated terminal pair and to be obtained port voltage or port current. First, let us write the port current:

\[
I_{ab} = \frac{R_2 U_i}{R_1 R_2 + (R_1 + R_2) R_x} = \frac{P}{\Delta + R_x \Delta_s}
\]

where, the numerator, the coefficient of $R_x$ and independent terms of $R_1$ in denominator correspond to $P, \Delta_s$ and $\Delta$ in (22), respectively. The Norton current is equal to limit of $I_{ab}$ as $R_1$ approaches zero:

\[
J_{nec} = \lim_{R_1 \to 0} I_{ab} = \frac{U_i}{R_1}
\]

This result can be directly obtained from (24), $P/\Delta$. Now, let us write the port voltage,

\[
U_{ab} = \frac{\frac{R_x R_1}{R_1 + \frac{R_x R_2}{R_2 + R_x}} U_i}{R_1 + \frac{R_x R_2}{R_2 + R_x}} = \frac{(R_2 U_i) R_x}{R_1 R_2 + (R_1 + R_2) R_x} = \frac{PR_x}{\Delta + R_x \Delta_s}
\]

where, the coefficient of $R_x$ in numerator, the coefficient of $R_x$ and independent terms of $R_1$ in denominator correspond to $P, \Delta_s$ and $\Delta$ in (23), respectively. As seen from the expressions of port current and port voltage, the components ($P, \Delta_s, \Delta$) of the method are the same. The Thévenin voltage is equal to limit of $U_{ab}$ as $R_1$ approaches infinite:

\[
U_{th} = \lim_{R_1 \to \infty} U_{ab} = \frac{R_2}{R_1 + R_2} U_i
\]

This result can be directly obtained from (26), $P/\Delta_s$. The Thévenin/Norton resistance can be obtained from the expressions of port current or voltage. The Thévenin/Norton resistance is equal to value of the test load resistance which drops the port voltage/current to half. If we extract the test load resistance from the port current expression,

\[
R_x = \frac{R_2 U_i - I_{ab} R_2}{I_{ab}(R_1 + R_2)} \quad \text{so,} \quad R_{th} = R_{nec} = \lim_{I_{ab} \to \frac{U_i}{2}} \frac{R_1 R_2}{R_1 + R_2}
\]
or if the test load resistance is extracted from the port voltage expression,

\[ R_t = \frac{U_{ab} R_2}{R_1 U_1 - U_{ab} (R_1 + R_2)} \]  

so, \[ R_{th} = \lim_{v_{ab} \to 0} R_t = \frac{R_1 R_2}{R_1 + R_2} \]

Thévenin/Norton resistance can be directly obtained from (29) or (33), \( \Delta t \Delta x \). The proposed method is systematical and unified. All parameters of the equivalent circuits are obtained simultaneously by using one topology. The conventional methods require different topologies to derive all parameters. For the Thévenin voltage and the Norton current, the open-circuit voltage and short-circuit current are calculated, respectively. Next, the equivalent resistance is calculated by setting all independent sources to zero. Using a test voltage source or a test current source is another method to obtain the parameters of Thévenin/Norton equivalent circuits (12). That method does not require to be set all independent sources to zero. But, it gives directly the Norton equivalent when the voltage source is used as the test source and the Thévenin equivalent when the current source is used as the test source. Therefore, it uses two topologies.

The proposed method in this paper is convenient for realizing experimentally. The Thévenin/Norton parameters are experimentally obtained by connecting a potentiometer to the port terminals a and b.

The ammeter in Fig. 5(a) gives the Norton current when the potentiometer is equal to zero. The voltmeter in Fig. 5(b) gives the Thévenin voltage when the potentiometer is equal to infinite (or not connect to the port). The Thévenin/Norton resistance can be obtained from the structures in Fig. 5(a) or (b). The equivalent resistance is equal to the potentiometer resistance which drops the port current from Norton current to one-half of it in Fig. 5(a) or the port voltage from Thévenin voltage to one-half of it in Fig. 5(b).

In order to obtain experimentally the parameters of Thévenin/Norton equivalent circuits by using the conventional methods, the short-circuit current as Norton current and the open-circuit voltage as Thévenin voltage are measured. The port resistance gives the Thévenin/Norton resistance when all independent sources are set to zero. Consequently the conventional methods use more than one topology and take a long time. The method described in Ref. (12) uses a test voltage source for Norton equivalent or a test current source for Thévenin equivalent without requiring setting all independent sources to zero. But, experimental application of the method is restricted because of nonexistence a physical current source. The methods presented in Refs. (11,16) are only experimental methods and do not include any equation.

Next three examples are presented to verify and evaluate the proposed method. In order to obtain the system equations, state variable analysis in the second example, mesh analysis in the third example and modified nodal analysis in the fourth example have been utilized. After connecting the test load impedance, \( Z_t \), into the output port of the circuit and expressing the port voltage or port current, the equivalent circuit parameters are obtained simultaneously.

Example 2. Consider the circuit shown in Fig. 6 where the test load impedance, \( Z_t \), is connected to the port terminals, a and b.

The state equations of the circuit in \( s \)-domain are:

\[
\begin{bmatrix}
    s U_c(s) \\
    i_t(s)
\end{bmatrix} =
\begin{bmatrix}
    0 & -\frac{1}{s} & \frac{1}{s} \\
    -\frac{1}{s} & -\frac{1}{s} + \frac{R_{ZX}}{s (s + R_{ZX})} & \frac{R_{ZX}}{R_L}
\end{bmatrix}
\begin{bmatrix}
    U_c(s) \\
    i_t(s)
\end{bmatrix} +
\begin{bmatrix}
    0 \\
    \frac{1}{1/s}
\end{bmatrix}
\begin{bmatrix}
    U_s(s)
\end{bmatrix}
\]

where, capacitor voltage (\( U_c(s) \)) and inductor current (\( i_t(s) \)) are the variables of the method. Let us rearrange the system equations as in the form of (13):

\[
\begin{bmatrix}
    s & -\frac{1}{s} & \frac{1}{s} \\
    \frac{1}{s} & -\frac{1}{s} + \frac{R_{ZX}}{s (s + R_{ZX})} & \frac{R_{ZX}}{R_L}
\end{bmatrix}
\begin{bmatrix}
    U_c(s) \\
    i_t(s)
\end{bmatrix} =
\begin{bmatrix}
    0 \\
    \frac{1}{1/s}
\end{bmatrix}
\begin{bmatrix}
    U_s(s)
\end{bmatrix}
\]

The determinant of the coefficient matrix is:

\[
\det(A) = \frac{Q(s)}{R(s)} = \frac{s R_{ZX} + s^2 R_L + s^2 L C + s R C Z_X + R + Z_t}{R_L (R + Z_t)}
\]

The characteristic equation, \( Q(s) \), is grouped according to \( Z_t \) as:

\[
Q(s) = \Delta(s) + Z_t \Delta_x(s) = [s^2 R_L + R] + Z_t [s^2 L C + s R C + 1]
\]

From the expression of \( Q(s) \) in (22) and (23), the components, \( \Delta \) and \( \Delta_x \), are determined as:

\[
\Delta(s) = s^2 R_L + R, \quad \Delta_x(s) = s^2 L C + s R C + 1
\]

After solving the system equations, the port voltage and the port current relating to the circuit are expressed as follows:

\[
L_{ab}(s) = \frac{R_1 U_i(s)}{R_1 + R_2 + Z_t}
\]

\[
U_{ab}(s) = \frac{R_1 U_i(s)}{R_1 + R_2 + Z_t}
\]

It can be easily understood that the denominators of the output variables are equal to the characteristic equation. Let’s rearrange the expressions of the port current and the port voltage
that the output voltage/current drops to half when the test load
approaches to one-half of its short circuit value (Iab)
or
Uab

The Norton current is equal to (24):

\[
I_{\text{N}} = \frac{P(s)}{sRCU_i(s)}
\]

The equivalent circuit parameters are determined in terms of
\(P(s)\) and the components of the characteristic equation. Actually,
it means that the limit of the port voltage as \(Z_t\) approaches to
infinity is equal to the Thévenin voltage:

\[
U_{\text{th}} = \lim_{Z_t \to \infty} U_{\text{ab}}(s) = \frac{P(s)}{sRC + R} = \frac{sRC}{s^2LC + sRC + 1} U_i(s)
\]

At the same time, the Thévenin voltage can be directly obtained
from the generalized expression of the Thévenin voltage given in
(26). The limit of the port current as the test load approaches to
zero gives the Norton current:

\[
J_{\text{N}} = \lim_{Z_t \to 0} I_{\text{ab}}(s) = \frac{P(s)}{s^2RCU_i(s)} = \frac{sRC}{s^2RC + R} U_i(s)
\]

The Norton current can be directly obtained from the generalized
expression of the Norton current given in (24).

Extracting the test load expression from the expressions of
\(U_{ab}\) or \(I_{ab}\), the below equations are obtained in terms of
\(U_{ab}\) and \(I_{ab}\), respectively:

\[
Z_t = \frac{s^2RLC + R}{sRCU_i(s) - s[2LC + sRC + 1]} U_{ab}(s)
\]

\[
Z_t = \frac{sRCU_i(s) - s[2LC + sRC + 1]}{I_{ab}(s)[s^2LC + sRC + 1]}
\]

The Thévenin/Norton impedance is equal to the limit of the
test load’s expression as the output voltage, \(U_{ab}\), approaches to
one-half of its open circuit value (Uoc) or the output current, \(I_{ab}\),
approaches to one-half of its short circuit value (IoC). It means
that the output voltage/current drops to half when the test load
impedance is equal to the Thévenin/Norton impedance:

\[
Z_{\text{th}} = Z_{\text{N}} = \lim_{V_{\text{ab}} \to \frac{1}{2} UOC} \lim_{I_{\text{ab}} \to \frac{1}{2} IOC} Z_t = \frac{\Delta(s)}{\Delta_X(s)} = \frac{s^2RLC + R}{s^2LC + sRC + 1}
\]

The result can be also obtained directly from (33), the
generalized expression of the Thévenin/Norton impedance.

Example 3: Consider the magnetic coupling circuit, shown in
Fig. 7 where the test load impedance, \(Z_t\), is connected to the port
terminals, a and b.

Mesh equations are: \(AX(s) = BU(s)\), where

\[
A = \begin{bmatrix}
sL_1 + sL_2 + 2sM & -sL_2 - sM - R & sM \\
-sL_2 - sM - R & sL_2 + 2sM & -sL_2 - sM \\
+ \frac{1}{RC} & - \frac{1}{RC} & - \frac{1}{RC} \\
\end{bmatrix}
\]

\[
X(s) = \begin{bmatrix} I_1(s) \\ I_2(s) \\ I_{ab}(s) \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad U(s) = [U_i(s)]
\]

The determinant of the coefficient matrix is:

\[
\det(A) = \frac{Q(s)}{R(s)} = \frac{\Delta(s) + Z_t \Delta_X(s)}{R(s)}
\]

The characteristic equation, \(Q(s)\), is grouped according to \(Z_t\) as:

\[
Q(s) = \Delta(s) + Z_t \Delta_X(s)
\]

where

\[
\Delta(s) = -s^3C[L_1(L_1 + L_2 + 2M) - M^2 + L_2L_3]
\]

\[
- s^2RC(L_1 + L_3) - s(L_1 + L_2)R
\]

\[
\Delta_X(s) = s^3L_1(C (M^2 - L_2L_3) - s^3RL_1L_3C)
\]

\[
- s^2L_1L_2 - sRL_1, \quad R(s) = sC
\]

After solving the system equations, the port current or port voltage
relating to the circuit is expressed. In this example, the port current
is used since the variables of mesh analysis are currents:

\[
I_{ab}(s) = \frac{P(s)}{Q(s)} = \frac{s^3C(M^2 - L_2L_3) - s^3RL_1C - sL_2 - R}{\Delta(s) + Z_t \Delta_X(s)} U_i(s)
\]

The parameters of the equivalent circuits are obtained by either
applying of the limit approach step by step as in Example 1 or
using the generalized expressions given in Section 3. In this
example and next example, the generalized expressions are used
in order to emphasis the systematic of the method.

The Thévenin voltage from (26) is as below:

\[
U_{ab}(s) = \frac{P(s)}{Q(s)} = \frac{s^3C(M^2 - L_2L_3) - s^3RL_1C - sL_2 - R}{\Delta(s) + \Delta_{11}(s)} U_i(s)
\]

where

\[
\Delta_{11}(s) = -s^3C[L_1(L_3 + L_2 + 2M) - M^2 + L_2L_3]
\]

\[
- s^2RC(L_1 + L_3) - s(L_1 + L_2)R
\]

The Norton current is equal to (24):

\[
J_{\text{N}}(s) = \frac{P(s)}{\Delta(s)} = \frac{[s^3C(M^2 - L_2L_3) - s^3RL_1C - sL_2 - R]}{s^2L_1L_2 - sRL_1, \quad R(s) = sC}
\]
The Thévenin/Norton impedance is equal to (33):

\[ Z_{th}(s) = Z_{nor}(s) = \frac{\Delta(s)}{\Delta_x(s)} = s^2L_4C(M^2 - L_3L_2) - s^2RL_1L_2 - s^2L_1L_2 - sRL \]

\[ \Delta_x(s) + \Delta_{12}(s) \]

Example 4: Consider the circuit shown in Fig. 8 where the test load impedance, \( Z_x \), is connected to the port terminals, a and b.

Modified nodal equations are: \( AX(s) = BU(s) \)

where

\[
A = \begin{bmatrix}
  G_2 + sC + 1/Z_x & -1/Z_x & -sC & -G_2 & 0 \\
  -1/Z_x & G_1 + 1/Z_x & 0 & 0 & 0 \\
  -sC & kG_1 & sC & 0 & 1 \\
  -G_2 & -kG_1 & 0 & \frac{1}{RL} + G_2 & 0 \\
  0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

\[
X(s) = \begin{bmatrix}
  U_a(s) \\
  U_b(s) \\
  U_c(s) \\
  U_d(s) \\
  I_a(s)
\end{bmatrix}, \quad B = \begin{bmatrix}
  0 & 0 \\
  1 & 0 \\
  0 & 0 \\
  -1 & 0 \\
  0 & 1
\end{bmatrix}, \quad U(s) = \begin{bmatrix}
  I_1(s) \\
  U_i(s)
\end{bmatrix}
\]

The determinant of the coefficient matrix is:

\[ \det(A) = \frac{Q(s)}{R(s)} = \frac{\Delta(s) + Z_x \Delta_x(s)}{R(s)} \]

The characteristic equation, \( Q(s) \), is grouped according to \( Z_x \) as:

\[ Q(s) = \Delta(s) + Z_x \Delta_x(s) \]

where

\[ \Delta_x(s) = s^2G_2G_1LC + sG_1C + G_1G_2, \]

\[ \Delta(s) = -s^2G_2LC - s(kG_1G_2L - G_1G_2L - C) + G_1 + G_2, \]

\[ R(s) = sLZ_x \]

After solving the system equations, the port current or port voltage relating to the circuit is expressed. In this example, the port voltage is used since the variables of nodal analysis are voltages.

\[ U_{ab}(s) = U_a(s) - U_b(s) = \frac{P(s)Z_x}{Q(s)} = \frac{P(s)Z_x}{\Delta(s) + Z_x \Delta_x(s)} \]

where

\[ P(s) = [-s^2G_2LC + s(kG_1G_2L - G_1G_2L - C) - G_2]I_i(s) + [sG_1C (1 + sG_2L)]U_i(s) \]

The Thévenin voltage from (26) is as below:

\[ U_{th}(s) = \frac{P(s)}{\Delta_1(s)} = \frac{-s^2G_2LC + s(kG_1G_2L - G_1G_2L - C) - G_2}{s^2G_1G_2LC + sG_1C + G_1G_2}I_i(s) \]

\[ + \frac{sG_1C (1 + sG_2L)}{s^2G_1G_2LC + sG_1C + G_1G_2}U_i(s) \]

The Norton current is equal to (24):

\[ J_{nor}(s) = \frac{P(s)}{\Delta(s)} = \frac{-s^2G_2LC + s(kG_1G_2L - G_1G_2L - C) - G_2}{s^2G_1G_2LC + sG_1C + G_1G_2}I_i(s) \]

\[ + \frac{sG_1C (1 + sG_2L)}{s^2G_1G_2LC + sG_1C + G_1G_2}U_i(s) \]

The Thévenin/Norton impedance is equal to (33):

\[ Z_{th}(s) = Z_{nor}(s) = \frac{\Delta(s)}{\Delta_x(s)} = \frac{-s^2G_2LC + s(kG_1G_2L - G_1G_2L - C) - G_2}{s^2G_1G_2LC + sG_1C + G_1G_2}I_i(s) \]

\[ + \frac{sG_1C (1 + sG_2L)}{s^2G_1G_2LC + sG_1C + G_1G_2}U_i(s) \]

5. Conclusions

A new efficient and systematic approach for determining parameters of the Thévenin and Norton equivalent circuits and its example applications have been presented in this paper. It allows all equivalent circuit parameters to obtain simultaneously from only one circuit topology. It is sufficient to connect test load impedance to the output of circuit and to express the port voltage or the port current by using any formulation method. As a result, the generalized expressions are formulated for obtaining systematically all parameters. The proposed approach is independent from the analysis method and can be applied to all possible active and passive circuit structures as already presented in the four example applications verifying the efficiency of the method explicitly. For future works, a computer program about computation of the parameters can be written by employing the proposed method.

References

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